Design of index assignment in vector-quantizers under channel errors

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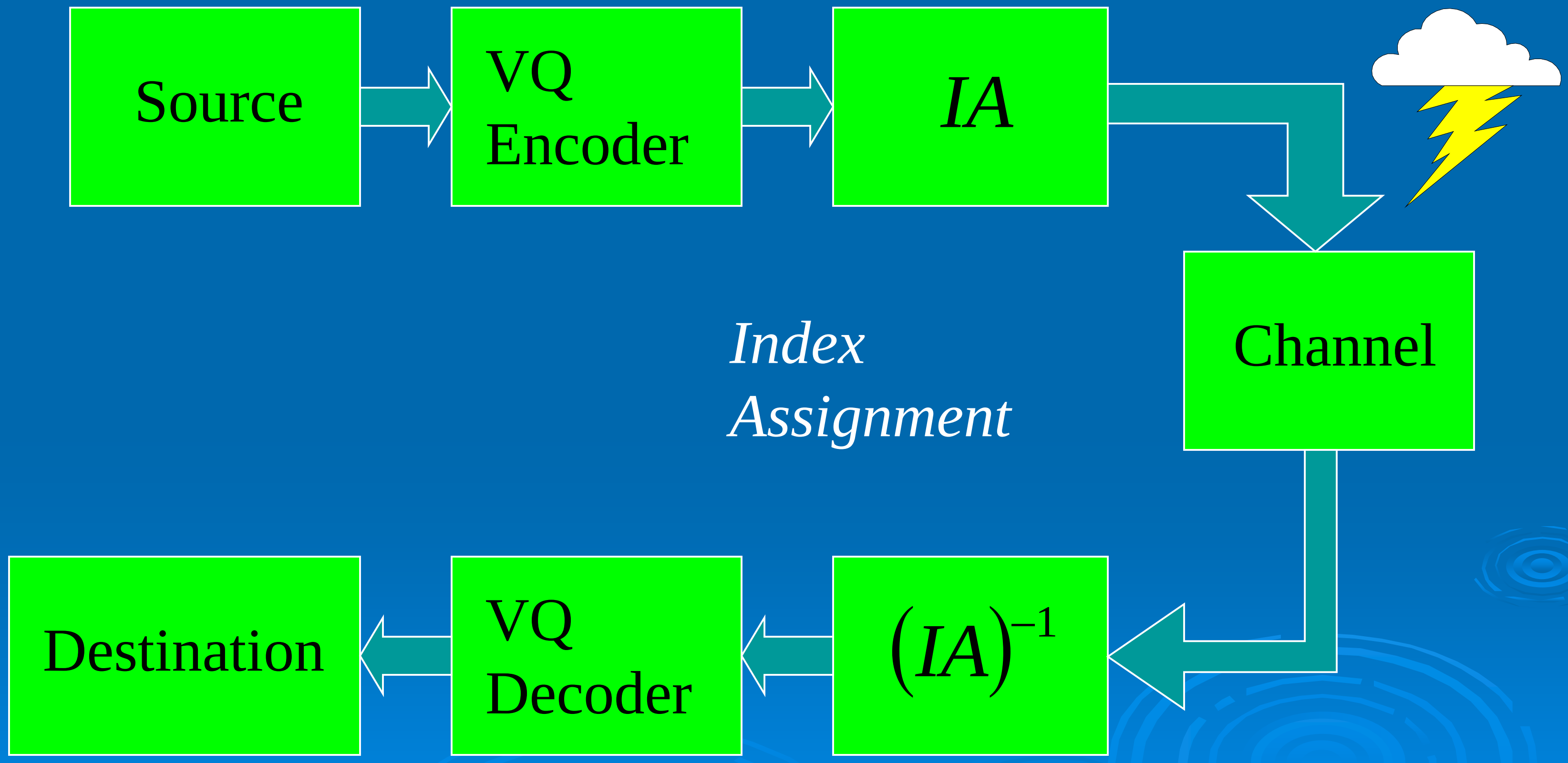
Outline

- Vector quantization
- Definition of channel distortion
- Index assignment
- Performance bounds
- Average performance over all index assignments
- Index assignment design algorithm
- Numerical results
- Conclusions

Vector Quantizer (VQ) Based Communication System

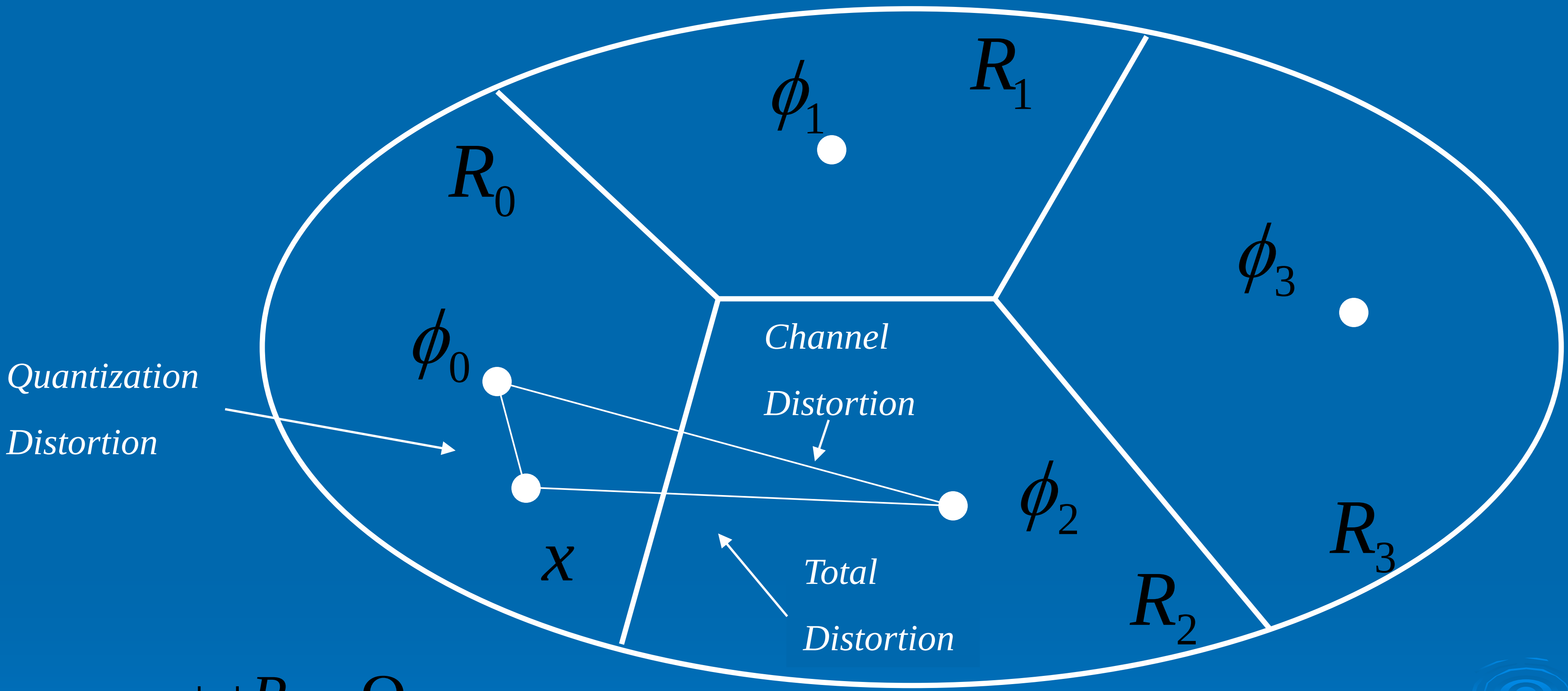


Vector Quantization

- Widely used method for low-bit-rate communication
- The *signal space* (Ω) of all possible source-vectors is divided into non-overlapping *regions* (R_i)
- Each region is represented by a *codevector* (ϕ_i)
- Codebook – The collection of all codevectors
- Codevector indices are sent through the channel

Signal Space

Ω - Entire Signal Space



$$\bigcup_i R_i = \Omega$$

$$R_i \cap R_j = \varnothing$$

R_i - Partition region i (of N)

ϕ_i - Codevector i

Distortion Values - I

$p(x)$ - Probability density function of source vector x

$d(\cdot,\cdot)$ - Distance Measure

π - Permutation Matrix

Q - Channel Transition Matrix

Average Quantization Distortion

$$\boldsymbol{D_{Q=I}} = \boldsymbol{E}\left[\boldsymbol{d}\left(\underline{\boldsymbol{x}},\underline{\hat{\boldsymbol{x}}}\right)\right]\Big|_{\boldsymbol{Q=I}} = \sum_{i=0}^{N-1}\int_{R_i}\boldsymbol{d}\left(\underline{\boldsymbol{x}},\underline{\phi}_i\right)\cdot\boldsymbol{p}\left(\underline{\boldsymbol{x}}\right)\cdot\boldsymbol{d}\,\underline{\boldsymbol{x}}$$

Total (overall) Average Distortion

$$\boldsymbol{D_T} = \boldsymbol{E}\left[\boldsymbol{d}\left(\underline{\boldsymbol{x}},\underline{\hat{\boldsymbol{x}}}\right)\right] = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{\pi\cdot\boldsymbol{Q}\cdot\pi^{T}\right\}_{ij}\int_{R_i}\boldsymbol{d}\left(\underline{\boldsymbol{x}},\underline{\phi}_j\right)\cdot\boldsymbol{p}\left(\underline{\boldsymbol{x}}\right)\cdot\boldsymbol{d}\,\underline{\boldsymbol{x}}$$

The partition regions and codevectors are designed to minimize the quantization distortion

Distortion Values - II

Average Channel Distortion

$$D_C = \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1} \left\{ \pi \cdot Q \cdot \pi^T \right\}_{ij} \cdot d\left(\underline{\phi}_i , \underline{\phi}_j \right) = trace\left\{ P \pi Q \pi^T D \right\}$$

where

$P = diag\left\{ p_0 , p_1 , \ldots , p_{N-1} \right\}$ - Partition regions probability matrix

$\left\{ D \right\}_{ij} = d\left(\underline{\phi}_i , \underline{\phi}_j \right)$ - codevectors distance matrix

For the Euclidean distance measure and Centroid Quantizers

Total distortion = Quantization distortion + Channel distortion

$$D_T = D_{Q=I} + D_C$$

Index Assignment

- Assignment of indices to codevectros affects system performance under channel errors
- There are *N*! possible index assignments
- Looking for the best assignment is a *Quadratic Assignment* problem and is known to be NP-complete
- Various <u>suboptimal, high complexity</u> index assignment algorithms are known – Local index switching, Genetic algorithms, Simulated annealing

Motivation for Determining Performance Bounds

- Difficulty in obtaining good assignments
- Need to estimate the performance of given assignments as compared to best and worst index assignments
- Basis for low complexity suboptimal index assignment algorithm

Bounds Outline - I

Average Channel Distortion

$$\boldsymbol{D_C} = \frac{1}{2}\,\boldsymbol{trace}\left\{\boldsymbol{Q}\pi^{\boldsymbol{T}}\hat{\boldsymbol{D}}\pi\right\} \quad \text{where } \hat{\boldsymbol{D}} = \boldsymbol{DP} + \boldsymbol{P}^{\boldsymbol{T}}\boldsymbol{D}^{\boldsymbol{T}}$$

Define

$$\boldsymbol{s_i} = \sum_{j=0}^{N-1} \hat{\boldsymbol{D}}_{\boldsymbol{ij}} \quad \text{and} \quad \boldsymbol{k} = arg\,max\left\{\boldsymbol{s_i}\right\}$$

$$\boldsymbol{C_i} = \begin{bmatrix} 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \end{bmatrix}$$

$\uparrow$

$\boldsymbol{i}$ – th column

Bounds Outline - II

Define

π - Permutation matrix

$$\alpha_i = (s_k - s_i)/N$$

$$S = \sum_{i=0}^{N-1} \alpha_i$$

$$\tilde{D} = \hat{D} + \sum_{i=0}^{N-1} \alpha_i \left(C_i + C_i^T\right)$$

λ_i - Eigenvalues of the channel transition matrix Q (descending order)

ω_i - Eigenvalues of the matrix $\tilde{D}$ (descending order)

and

$$Q = V \cdot \Lambda \cdot V^T \quad V \cdot V^T = I$$

$$\tilde{D} = W \cdot \Omega \cdot W^T \quad W \cdot W^T = I$$

$$\Psi = V^T \pi^T W$$

Bounds Outline - III

$$D_c = \frac{1}{2} trace\left\{V \cdot \Lambda \cdot V^T \pi^T W \cdot \Omega \cdot W^T \pi\right\} - S =$$

$$= \frac{1}{2} trace\left\{\Lambda \cdot V^T \pi^T W \cdot \Omega \cdot W^T \pi V\right\} - S =$$

$$= \frac{1}{2} trace\left\{\Lambda \cdot \Psi \cdot \Omega \cdot \Psi^T\right\} - S =$$

$$= \frac{1}{2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \lambda_i \omega_j \Psi_{ij}^2 - S$$

Bounds Outline - IV

We relax the constraint $\Psi = V^T \pi^T W$

to Ψ *unitary*

$$\min_{\Psi} / \max_{\Psi} \left(\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \lambda_i \omega_j \Psi_{ij}^2 \right)$$

$$s.t. \sum_{i=1}^{N-1} \Psi_{ij}^2 = 1 \quad j = 1, 2, \ldots, N-1$$

$$\sum_{j=1}^{N-1} \Psi_{ij}^2 = 1 \quad i = 1, 2, \ldots, N-1$$

Bounds Outline - V

Lower and upper bound are obtained using LP arguments

$$\text{Minimum value}: \sum_{i=1}^{N-1} \lambda_i \cdot \omega_{N-i}$$

Corresponding to:

$$\Psi_{\min} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & 1 \\ \vdots & & & 1 & \\ 0 & & \iddots & & \\ 0 & 1 & & & 0 \end{bmatrix}$$

$$\text{Maximum value}: \sum_{i=1}^{N-1} \lambda_i \cdot \omega_i$$

Corresponding to:

$$\Psi_{\max} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & \\ 0 & 0 & & & 1 \end{bmatrix}$$

Bounds Outline - VI

Lower and Upper Bounds

$$\frac{1}{2}\left(\lambda_0\omega_0+\sum_{i=1}^{N-1}\lambda_i\cdot\omega_{N-i}\right)-S\le D_C\le\frac{1}{2}\left(\lambda_0\omega_0+\sum_{i=1}^{N-1}\lambda_i\cdot\omega_i\right)-S$$

Complexity – Finding the eingenvalues of two matrices

Suboptimal assignment

Using LP obtain permutation p to approximate the lower bound scenario

$$\Psi_{\min} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & 1 \\ \vdots & & & 1 & \\ 0 & & \iddots & & \\ 0 & 1 & & & 0 \end{bmatrix} \approx V^T \pi^T W$$

$$R = V \Psi_{\min} W^T$$

$$\min_{\pi} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left(1 - r_{ij}^2\right) \pi_{ij}$$

$$s.t. \sum_{j=0}^{N-1} \pi_{ij} = 1 \quad i = 0,1,\ldots,N-1$$

$$\sum_{i=0}^{N-1} \pi_{ij} = 1 \quad j = 0,1,\ldots,N-1$$

$$\pi_{ij} \geq 0$$

Average Performance

A related expression for the ***average performance over all possible index assignments***:

$$\left\langle \boldsymbol{D}_{\boldsymbol{C}} \right\rangle = \frac{1}{2}\lambda_0\omega_0 + \frac{1}{2\left(\boldsymbol{N}-1\right)}\sum_{\boldsymbol{i}=1}^{\boldsymbol{N}-1}\sum_{\boldsymbol{j}=1}^{\boldsymbol{N}-1}\lambda_{\boldsymbol{i}}\omega_{\boldsymbol{j}} - \boldsymbol{S}$$

May also help in finding how well a given assignment performs

Special Cases and Numerical Results

- The proposed bounds were compared to the average performance as well as to “good“ and “bad” assignments found in simulations
- For 3-bit quantizers, all assignment were checked by exhaustive search
- For 4-bit and larger quantizers, “Good” (“bad”) assignments were found by a index switching algorithm (local optimization)

Uniform Scalar Quantizer and a Uniform Source Under the Binary Symmetric Channel (BSC)

$$\frac{2(N-1)(N+1)}{3N^2}2q \le D_C \le \frac{2(N-1)(N+1)}{3N^2}\left[1-(1-2q)^L\right]$$

$$\langle D_C \rangle = \frac{2N(N+1)}{3N^2}\left[1-(1-q)^L\right]$$

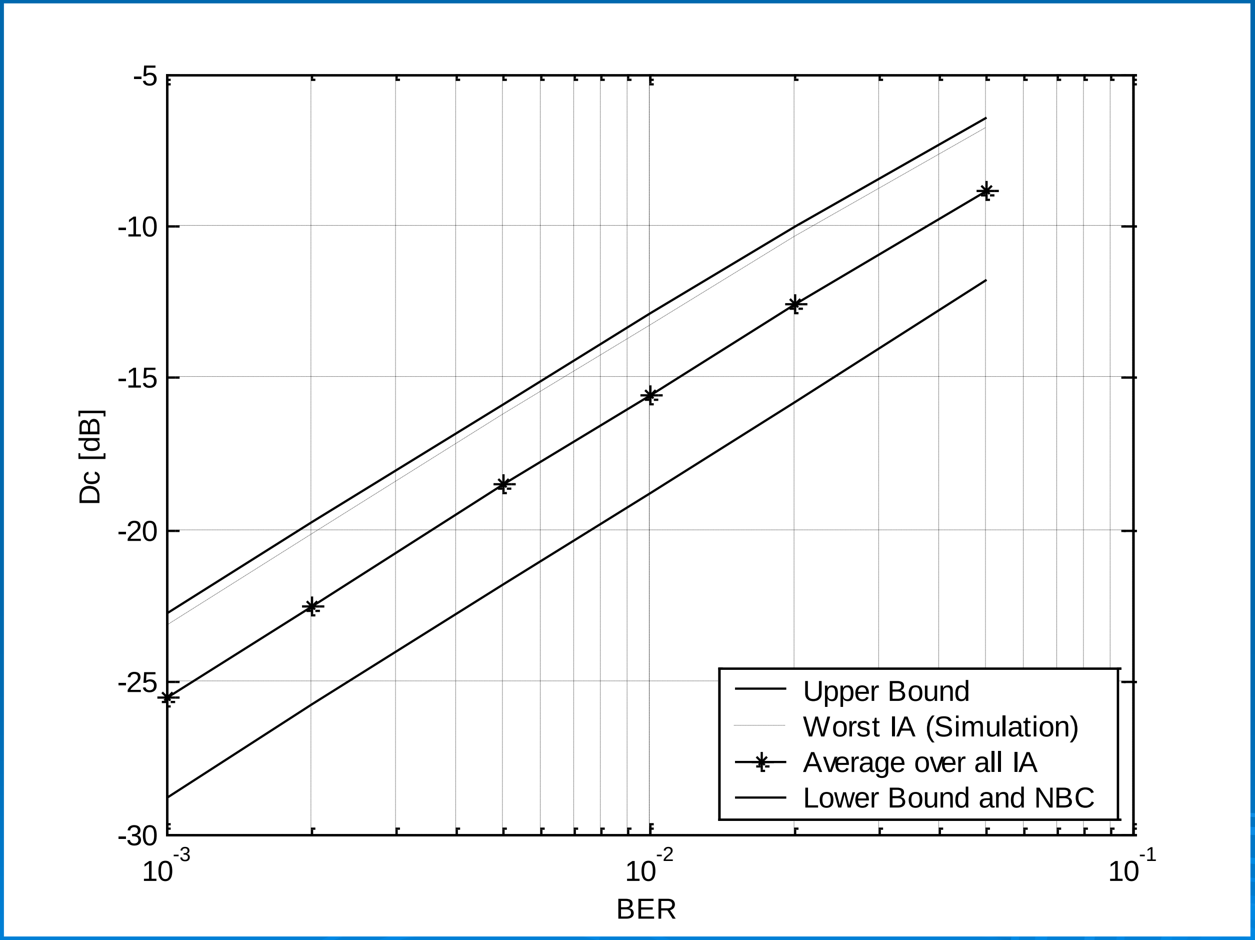
where

N - # of quantization levels

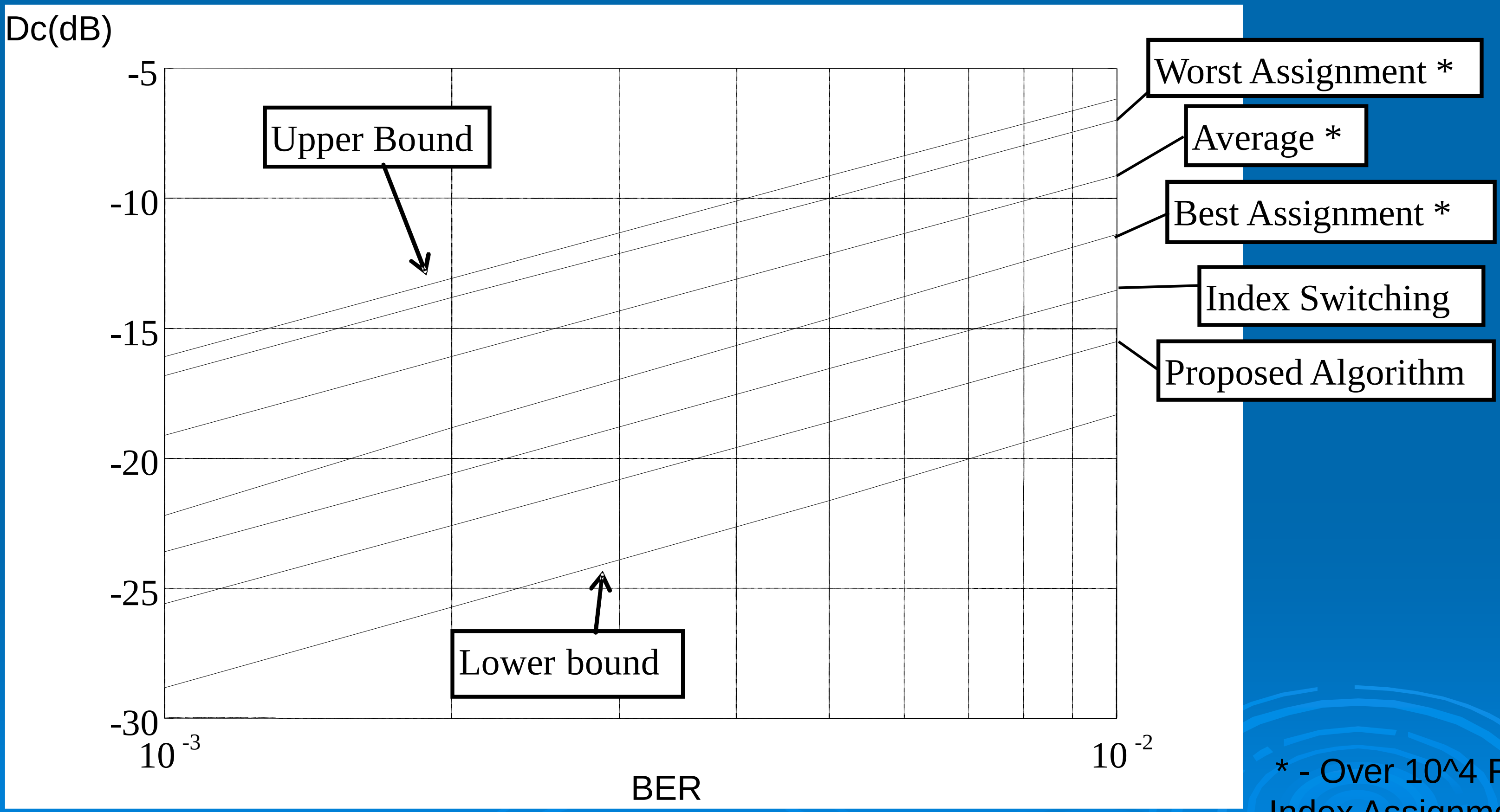
q - Bit Error Rate (BER)

The lower bound coincide with the performance of the Natural Binary Code

4-bit Uniform Scalar Quantizer and a Uniform Source Under the BSC

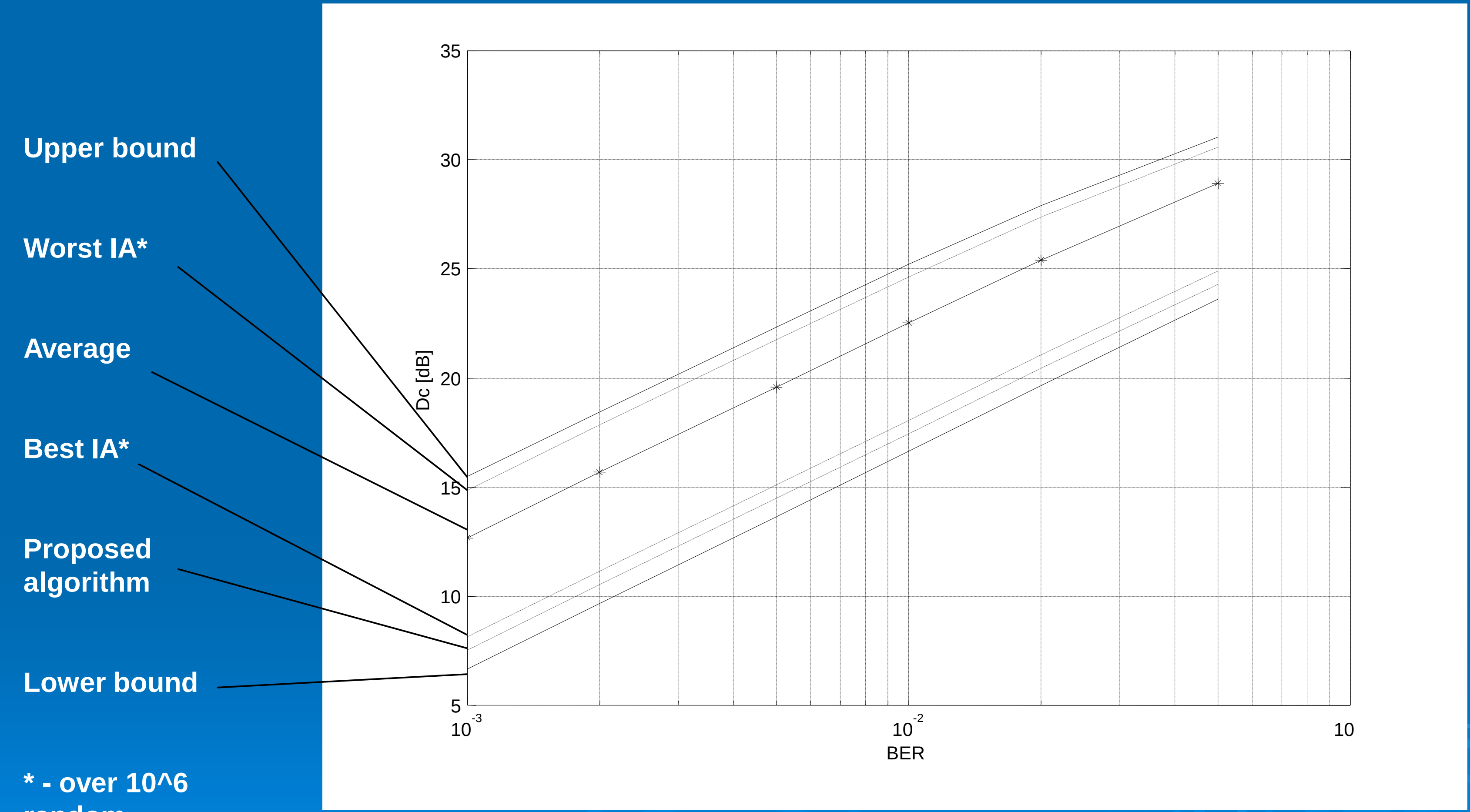


4-bit PDF optimized Uniform Scalar Quantizer and a Gaussian source under the BSC



* - Over 10^4 Random Index Assignments

Three-Dimensional, 8-bit PDF-Optimized Vector Quantizer for Palette Limited Images Using the L*a*b* Color Space



* - over 10^6 random assigments

Conclusions

- For vector quantizers operating under channel errors, upper and lower bounds on the average distortion, over all index assignments, were introduced.
- Related expression for the average performance, over all possible index assignments, was shown.
- Suboptimal low complexity IA algorithm was proposed. In simulations, the algorithm performs better than “index switching”.
- Bounds are reasonably close to the performance of the assignments found in simulations.