Technion - EE Department

Signal and Image Processing Lab.

Rate, Distortion, and Complexity Tradeoffs in Fractal Image Coding

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Talk Layout

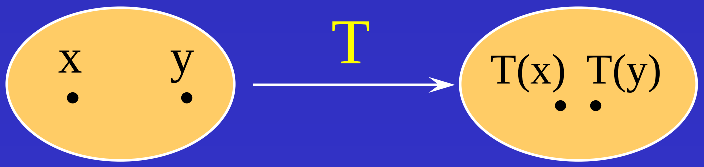
- Mathematical background.
- Fractals, what are they ?
- Fractal image coding - overview.
- Image partitioning and splitting criteria.
- Reducing encoder complexity.
- Fractal image coding combined with Matching - pursuit and VQ.
- Entropy coding of the “Fractal code”.
- Summary and conclusions.

Contractive Transformation

Let M be a metric space with metric d.

T is said to be a Contractive transformation ,

T:(M,d)→(M,d) , iff



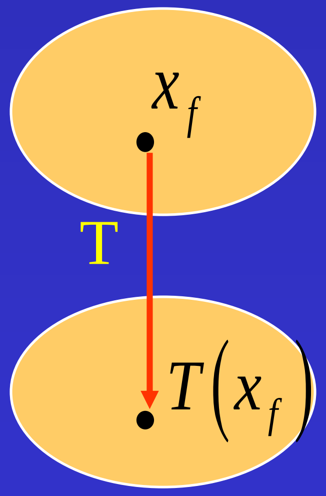
$$\exists s \quad 0 \le s < 1 \quad , \quad \forall x, y \in M \qquad d\big(T(x), T(y)\big) \le s d(x, y)$$

Fixed Point

Let (M,d) be a complete metric space and T:(M,d)→(M,d) be a contractive transformation.

Then there exists a unique point, (Fixed point) such that :

$$T\left(x_f\right) \to x_f$$



$$\forall x_o \in M, \quad T^n\left(x_o\right) \xrightarrow[n\to\infty]{} x_f$$

Collage Theorem

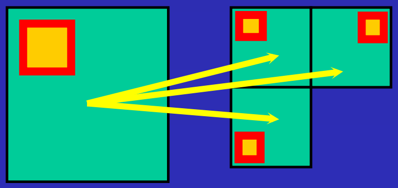
Let (M,d) be a complete metric space and T:(M,d)→(M,d) be a contractive transformation with a fixed point x_f .

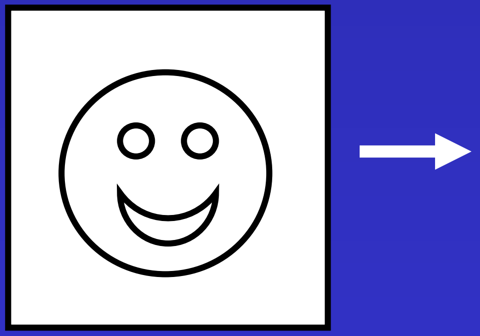
$$d\left(x, x_f\right) \leq \frac{1}{1-s} d\left(T\left(x\right), x\right)$$

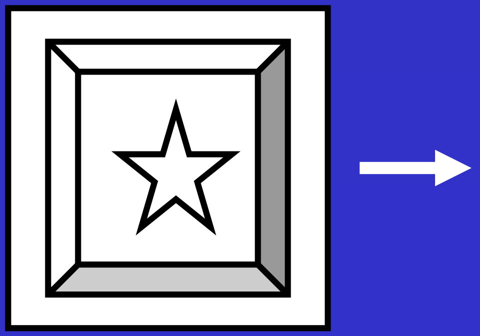
The inverse problem

Find T such that $T\left(x\right) \approx x_f$

Fractal Generation







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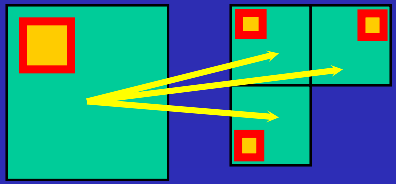

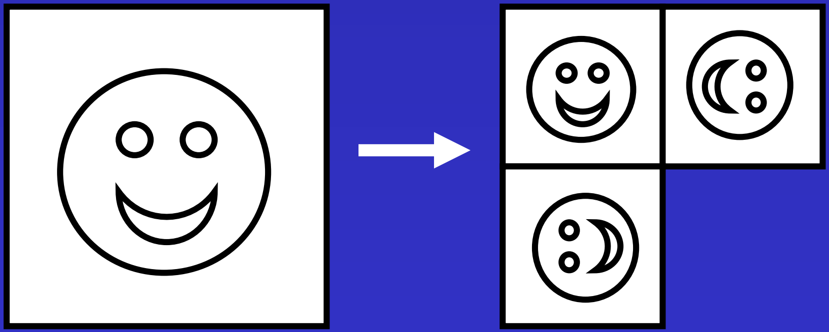

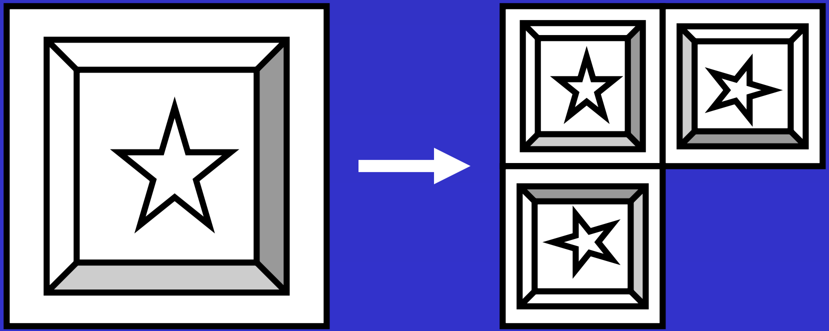

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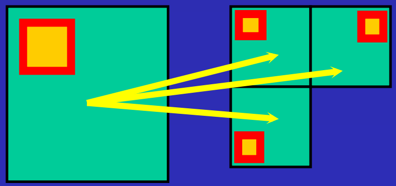
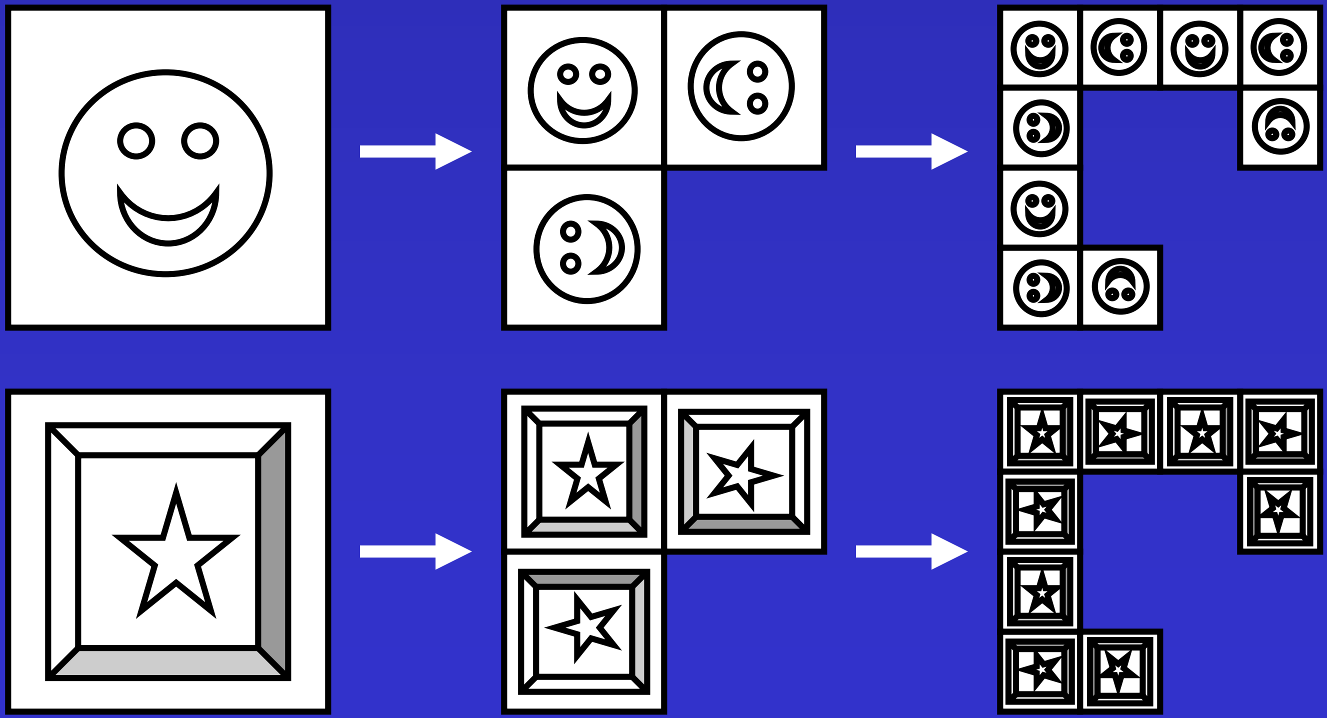
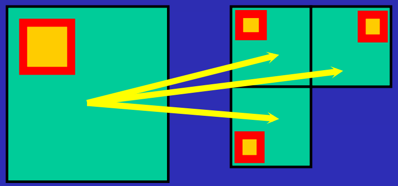
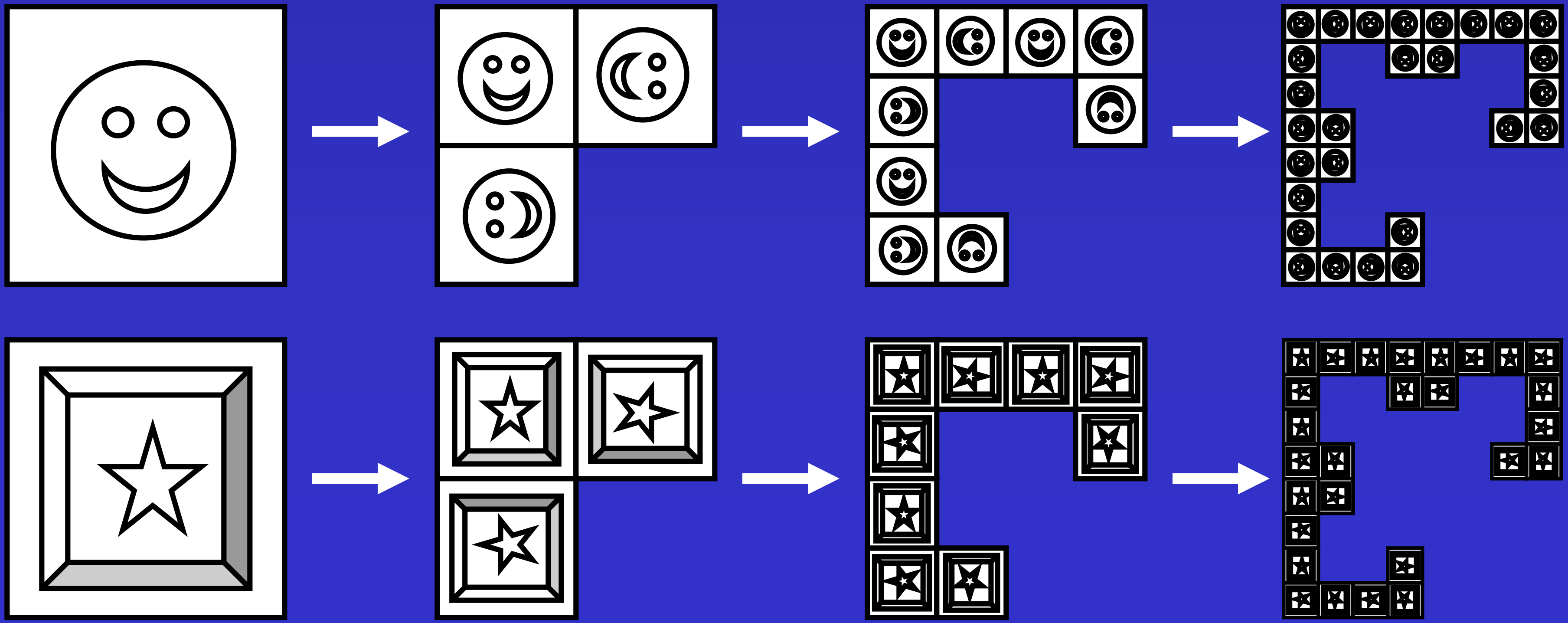


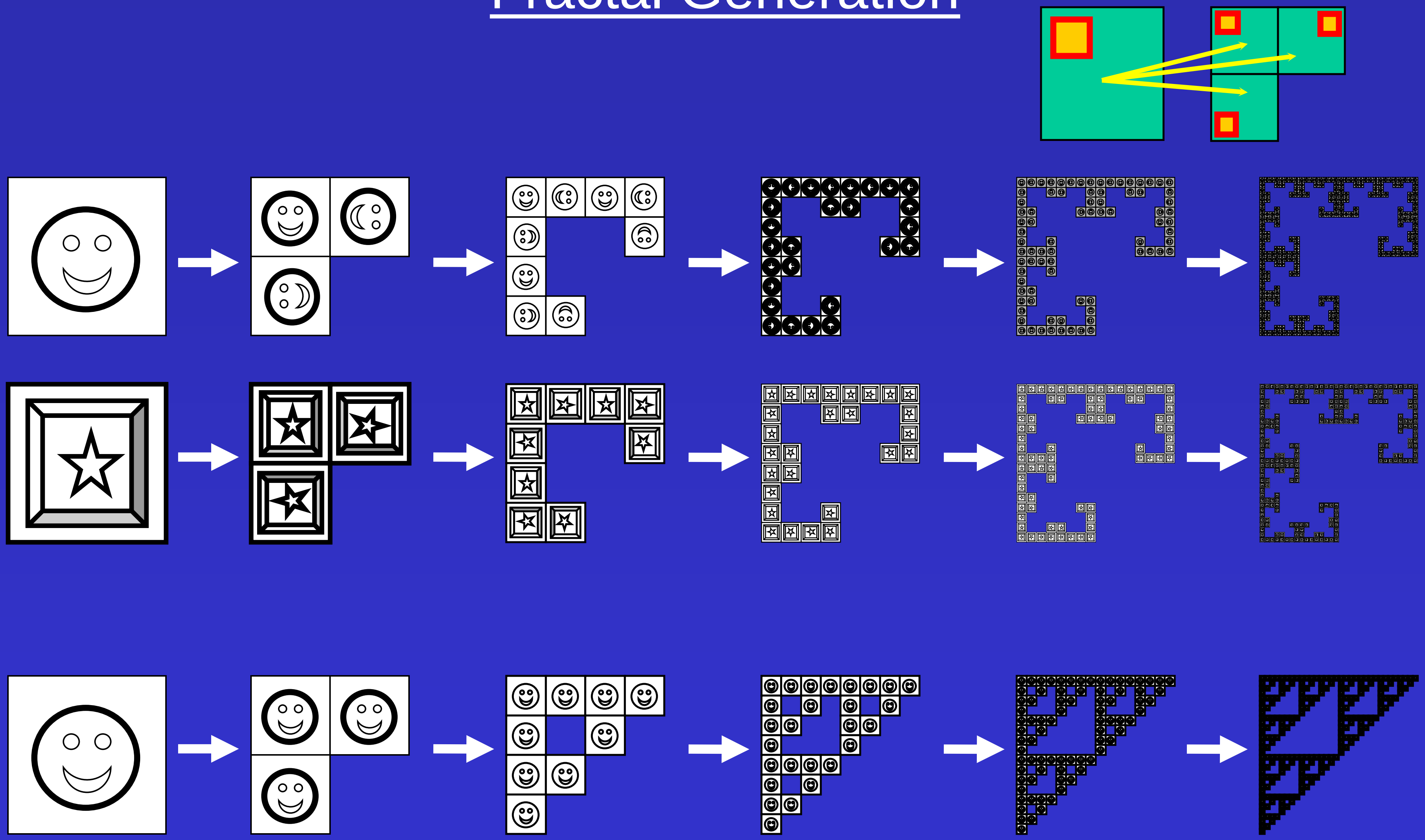


Fractal Generation



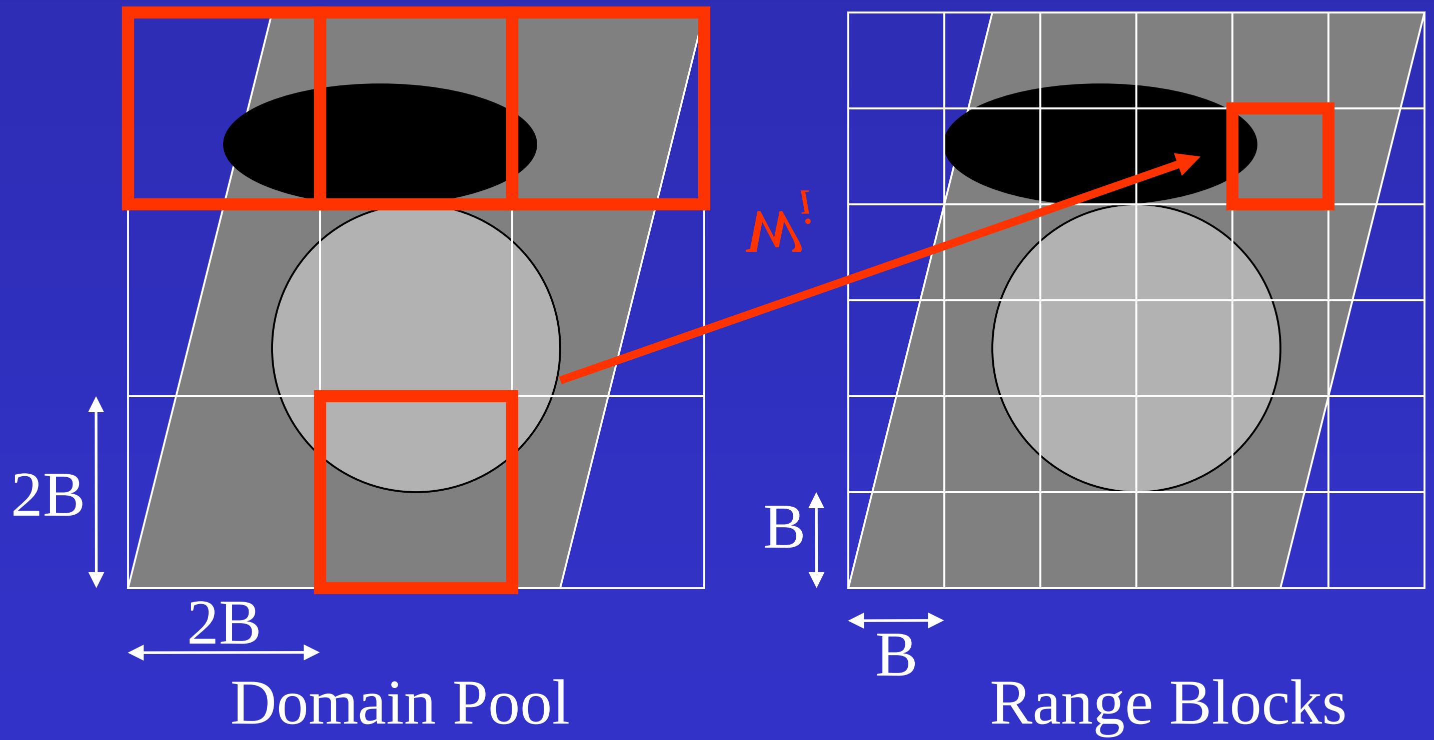


Fractal Generation



The First Fractal Image Coding Algorithm [Jacquin 1989]

Based on collage theorem : Find transformation such that its fixed point X_f is "close" to an image X.



For each range block, find best domain block, using an Affine transformation, such that minimize the Collage error

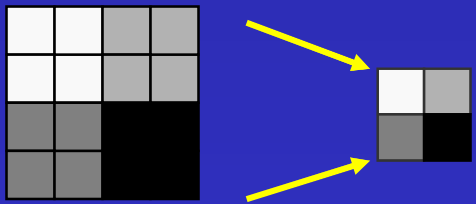
The First Fractal Image Coding Algorithm (cont'd)

[Jacquin 1989]

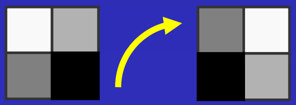
$$\hat{R}_i = a_i \cdot I_i\left(\phi\left(D_{j,i}\right)\right) + b_i \cdot 1^{B\times B} \qquad \min \left\| R_i - \hat{R}_i \right\|_2^2$$

Where:

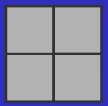
$\phi\left(D_{j,i}\right)$



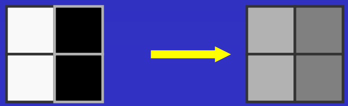
I_i



$b_i \cdot 1^{B\times B}$



a_i



- (w$_i$ is contractive if $|a_i| < 1$)

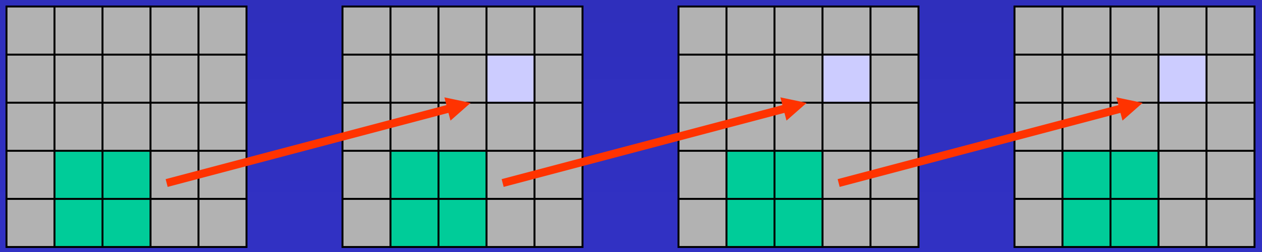
The "Fractal Code" to be transmitted : $W = \bigcup_i w_i = \bigcup_i \{a_i, j_i, b_i, I_i\}$

The First Fractal Image Coding Algorithm (cont'd)

[Jacquin 1989]

Iterative Decoding

Apply W iteratively to any initial image until successive iterations differs slightly



DC - Orthogonalization [Øien 1991]

Instead of using: $R_i \approx a_i \cdot \varphi\left(D_{j_i}\right) + b_i \cdot 1_{BxB}$

Use: $R_i \approx s_i \cdot \varphi\left(D_{j_i} - \overline{D}_{j_i}\right) + \overline{R}_i \cdot 1_{BxB}$

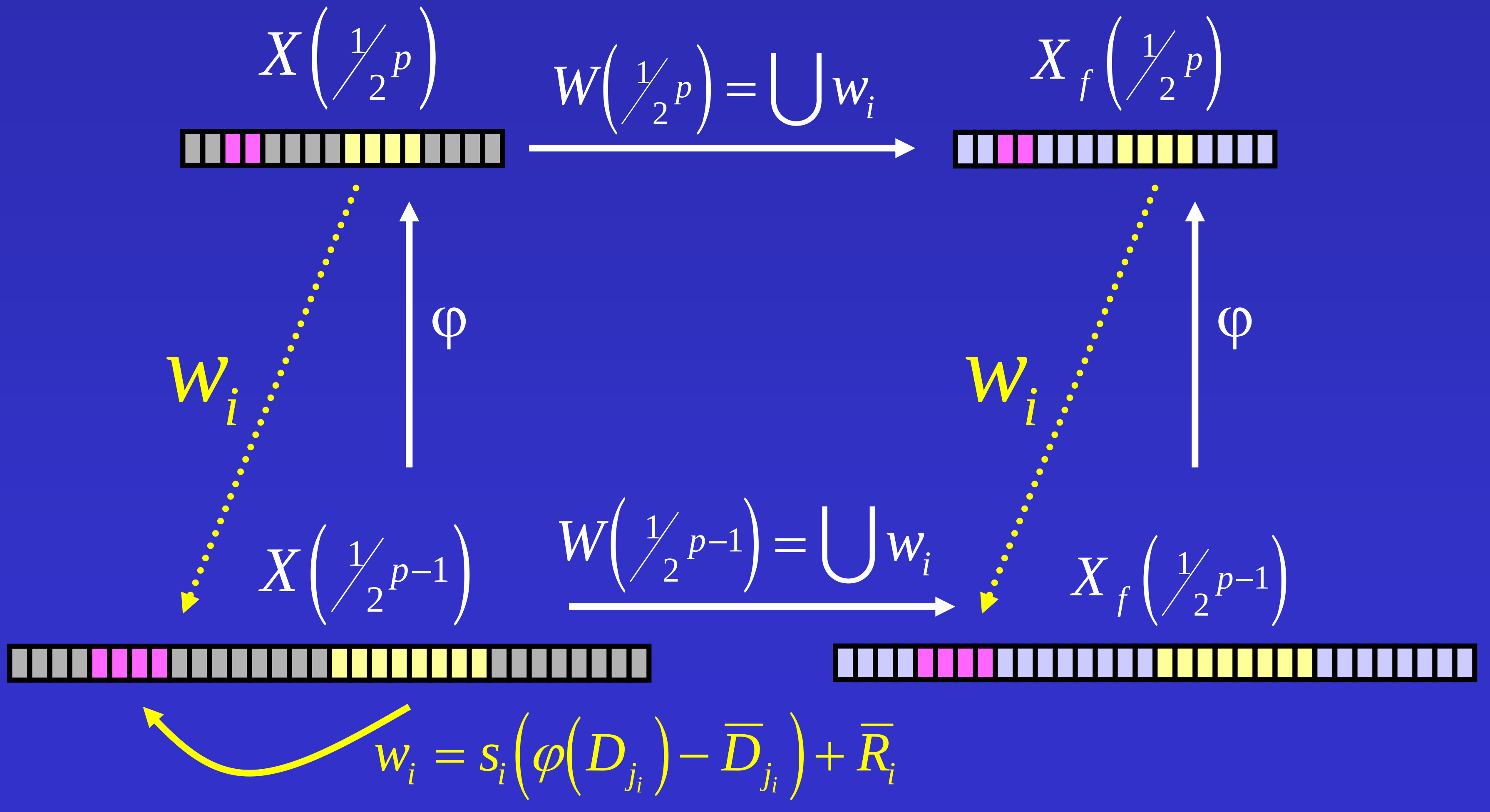
Where:

$\overline{R}_i$ - mean of range block

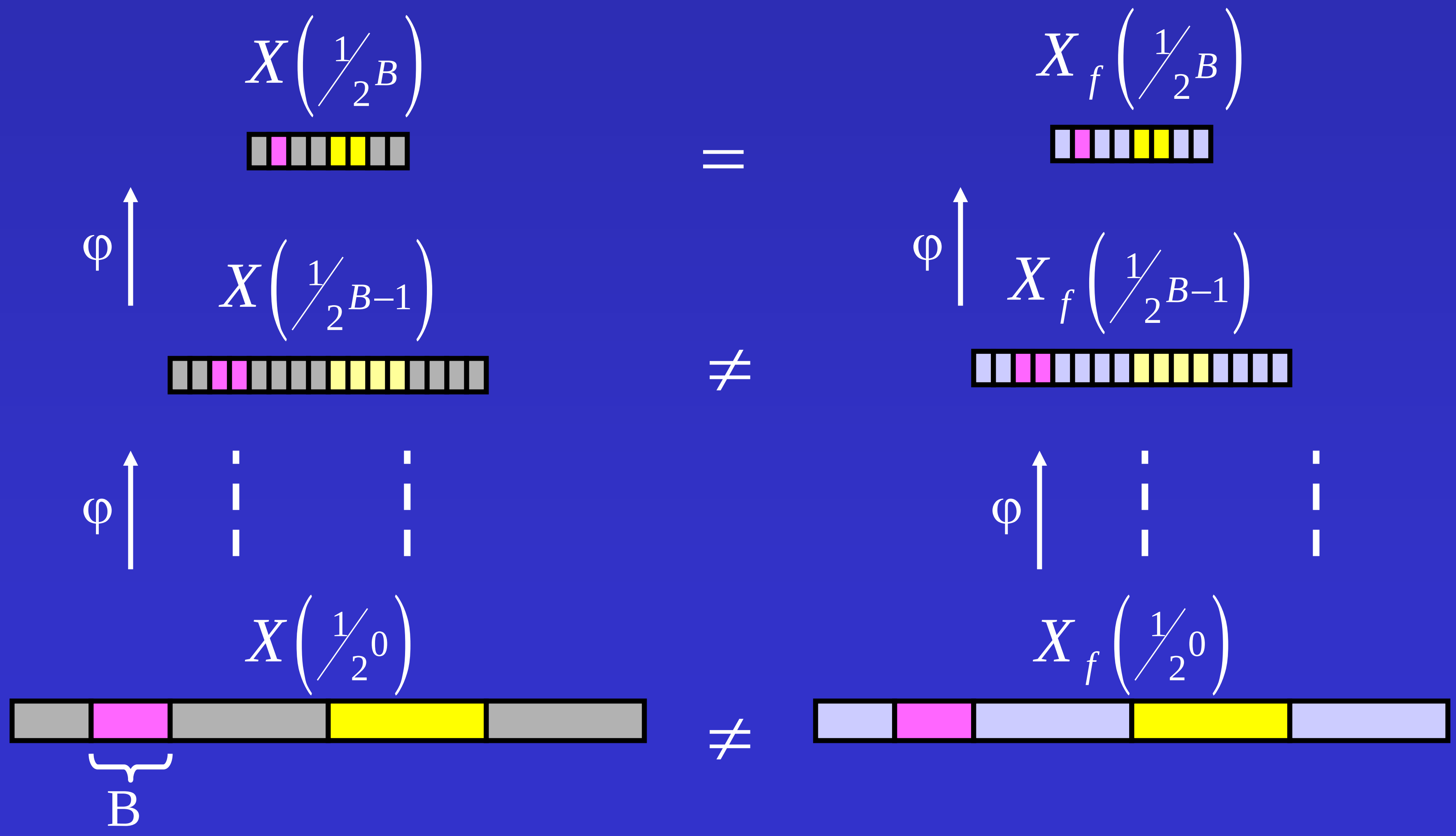
$\overline{D}_{j_i}$ - mean of domain block

The fractal code: $W = \bigcup_i w_i = \bigcup_i \left\{ s_i ,\ j_i ,\overline{R}_i \right\}$

Hierarchical Fast Decoding Combined with DC Orthogonalization [Baharav et al. 1993]



Hierarchical Fast Decoding Combined with DC Orthogonalization [Baharav et al. 1993] (cont'd)



Hierarchical Fast Decoding Combined with DC Orthogonalization [Baharav et al. 1993] (cont'd)

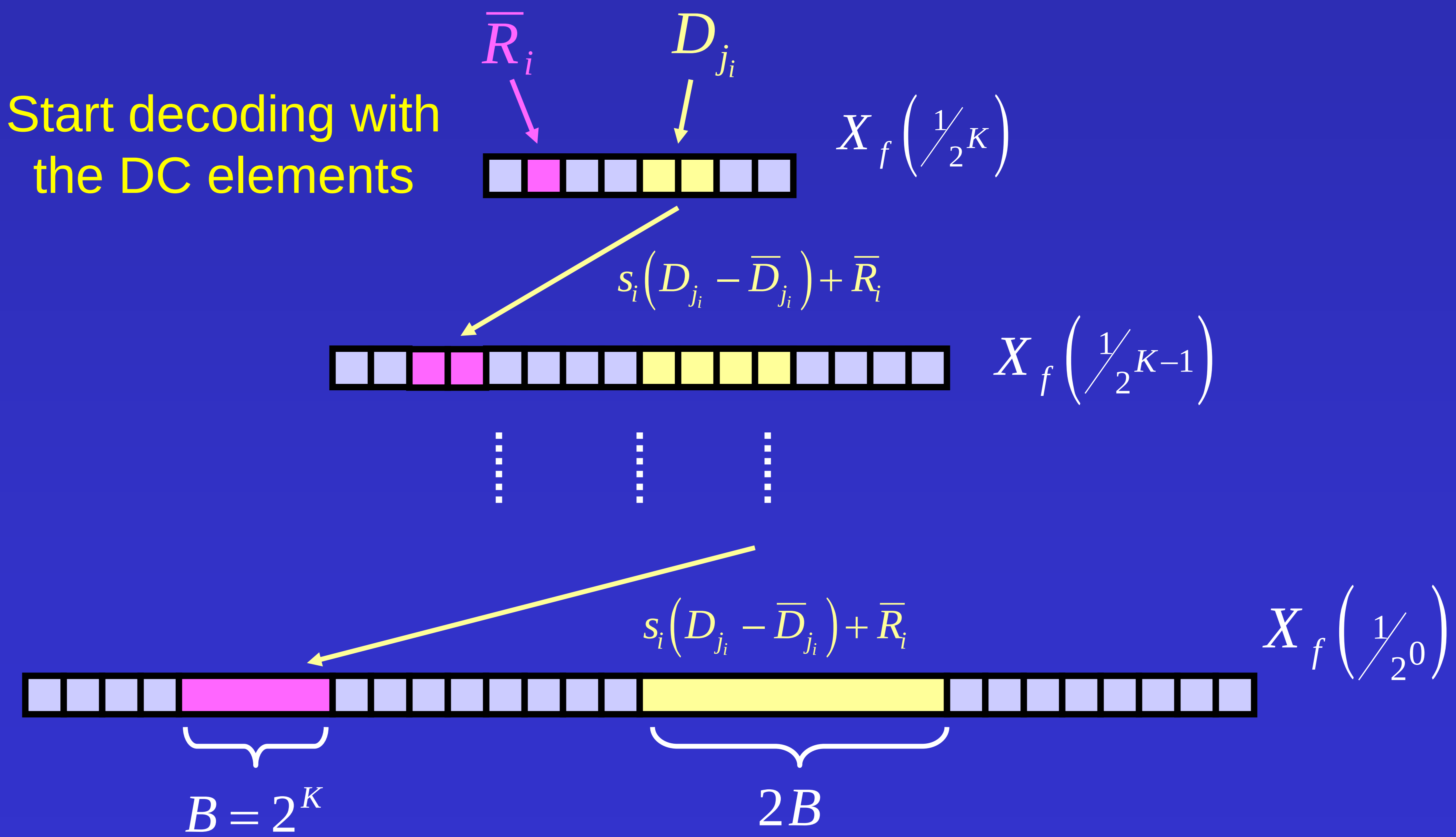
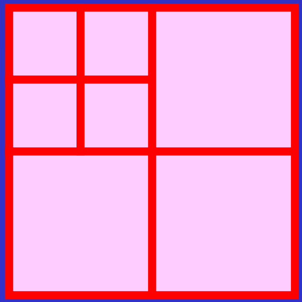


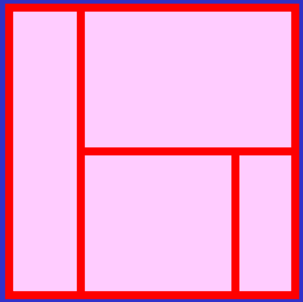
Image Partition and Split criteria

Using range blocks at various shapes and sizes improves coding efficiency.

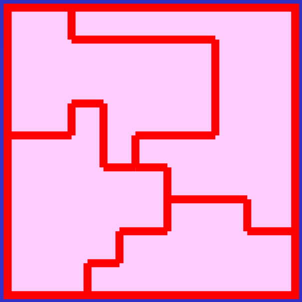
- Use top-down or bottom-up ? Merge small blocks to larger ones or split large blocks into small ones ?
- Which partitioning structure to use ? Quadtree, Triangles, HV...
- Which splitting criterion to use ? Entropy, Variance ...



Quadtree partitioning - Fisher 1992



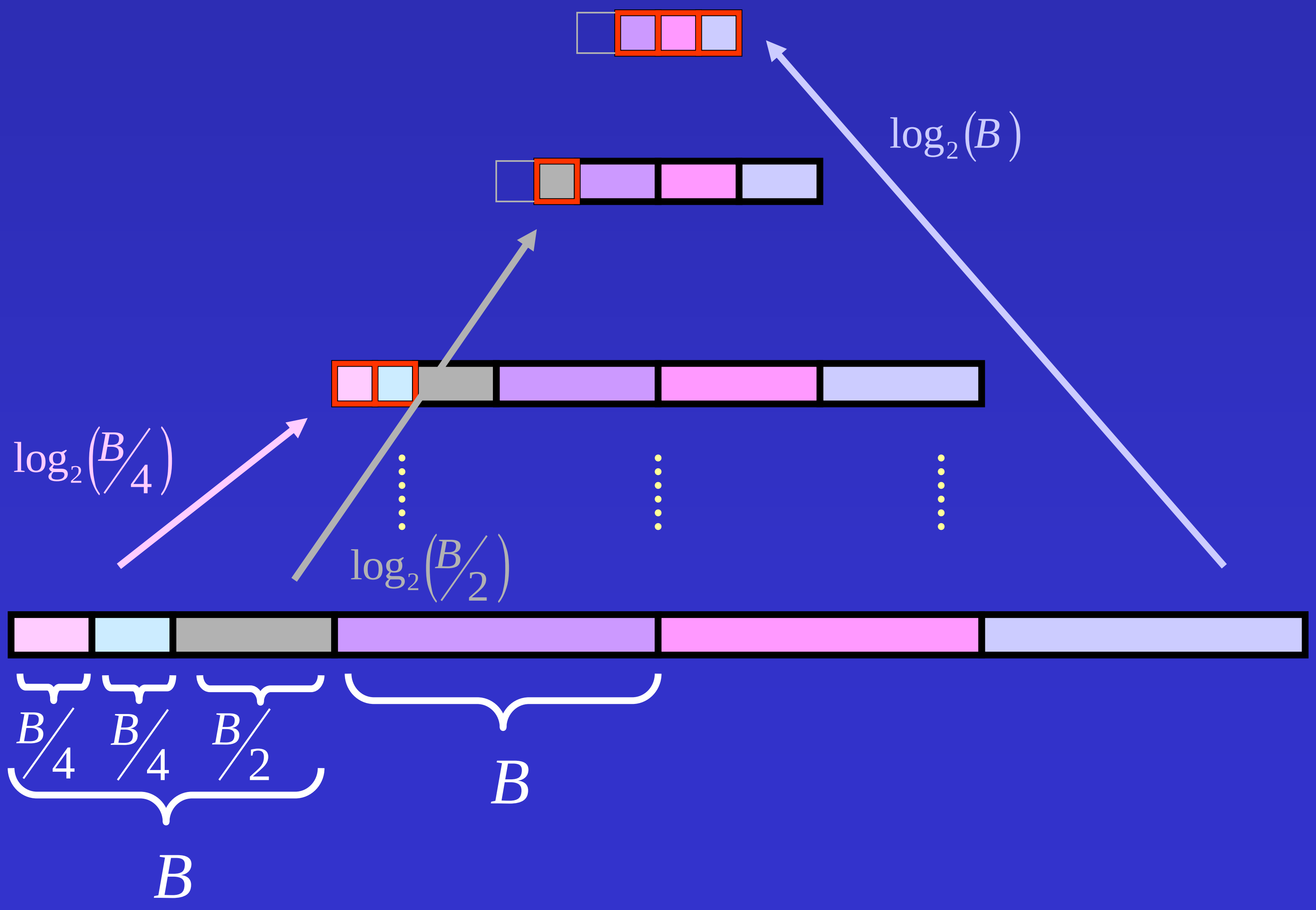
HV partitioning - Fisher 1995



Block merging (bottom-up) - Thomas 1995

Hierarchical Fast Decoding Using Quadtree Partitioning

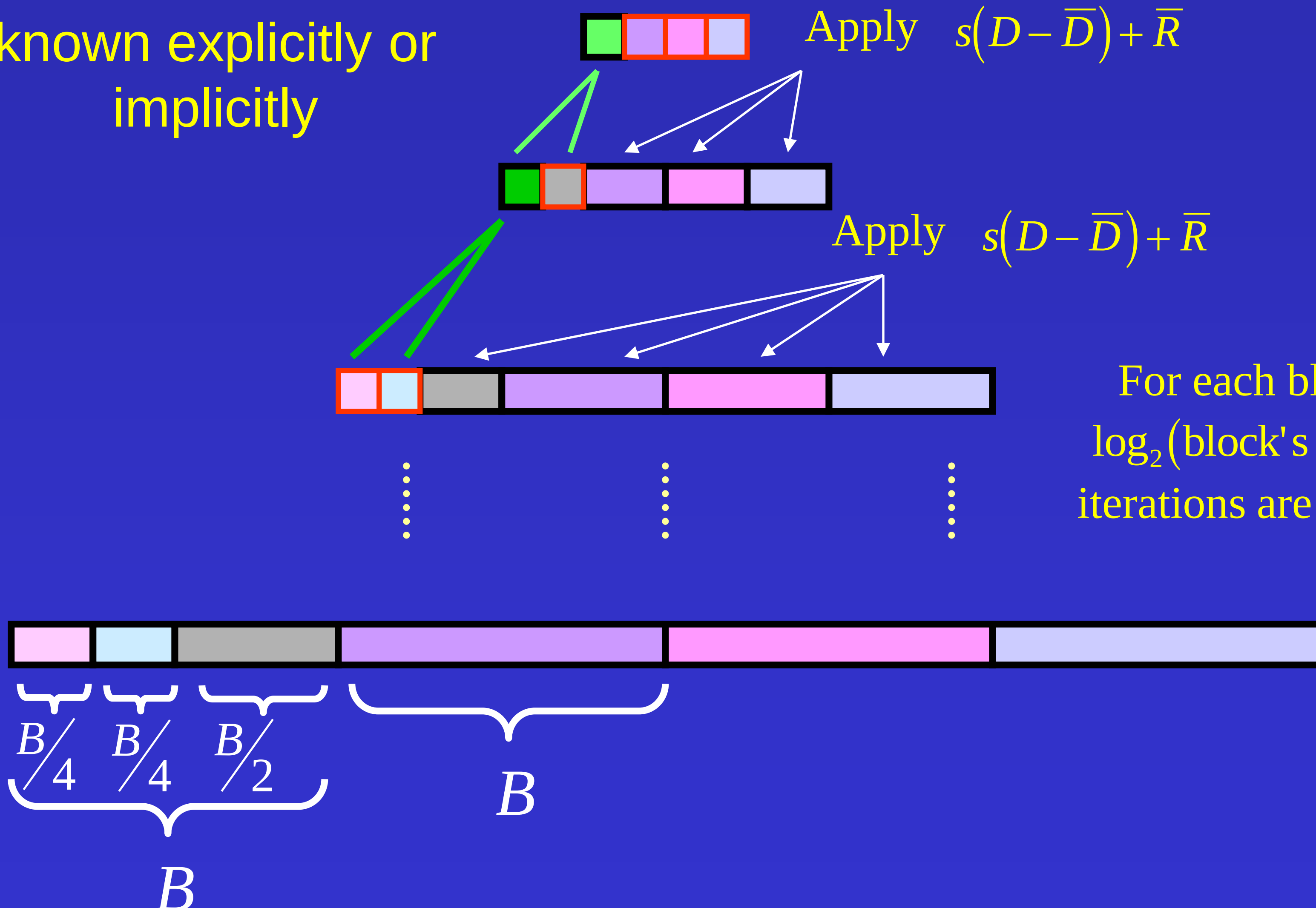
[Sutskover 1998]



Hierarchical Fast Decoding Using Quadtree Partitioning (cont'd)

[Sutskover 1998]

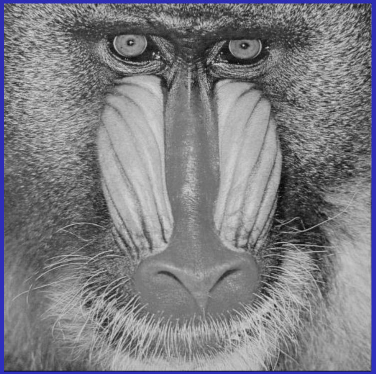
All mean values are known explicitly or implicitly



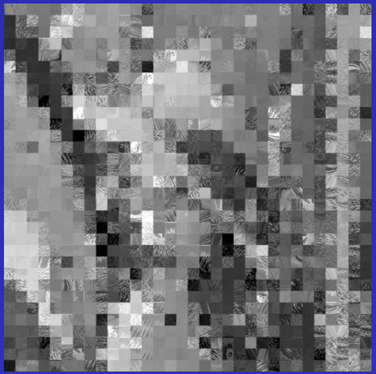
For each block, $\log_2\left(\text{block's size}\right)$ iterations are applied

Comparison of Decoding Algorithms

Iterative Decoding



Start image



1st iteration

2nd iteration

3rd iteration

- Infinite number of iterations are needed to converge to a fixed point

Comparison of Decoding Algorithms (cont'd)

Hierarchical Decoding

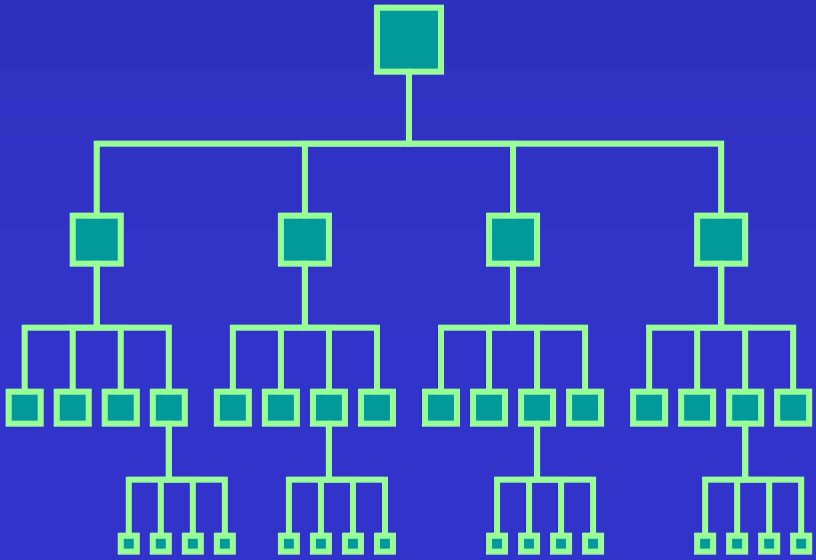
- $\log_2(\text{max block size})+1$ pyramid levels - finite, known, number of operations

- Each pyramid level contains less elements - reduction in computations
- True fixed point is achieved
- There are no contractivity considerations

Quadtree Image Partitioning

Threshold - based splitting criterion

Each range block is divided into four sub-range blocks if its collage error is less than a predefined threshold.



Quadtree Image Partitioning

Rate-Distortion - based splitting criterion

Gain G_{RDi}, for range block $\tilde{R}_i$, is defined as :

The Gain denotes the Collage error decrease per bit if a block is split

$$G_{RDi} = \frac{-\left(\left(\sum_{m=1}^{4} \left\| E_{i_m} \right\|_2^2\right) - \left\| E_i \right\|_2^2\right)}{\left(\sum_{m=1}^{4} \left| w_{i_m} \right|\right) - \left| w_i \right| + \left| Q_i \right|}$$

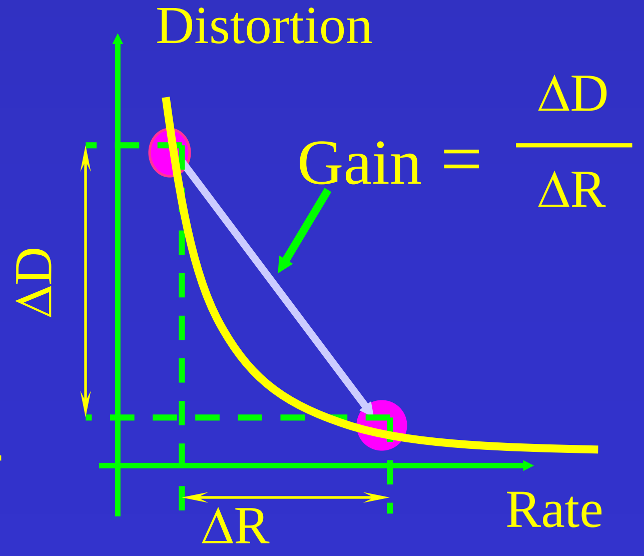
Where:

$$w_i = \{\bar{R}_i, j_i, s_i\}$$

$$|w_i| = |\bar{R}_i| + |s_i| + |j_i|$$

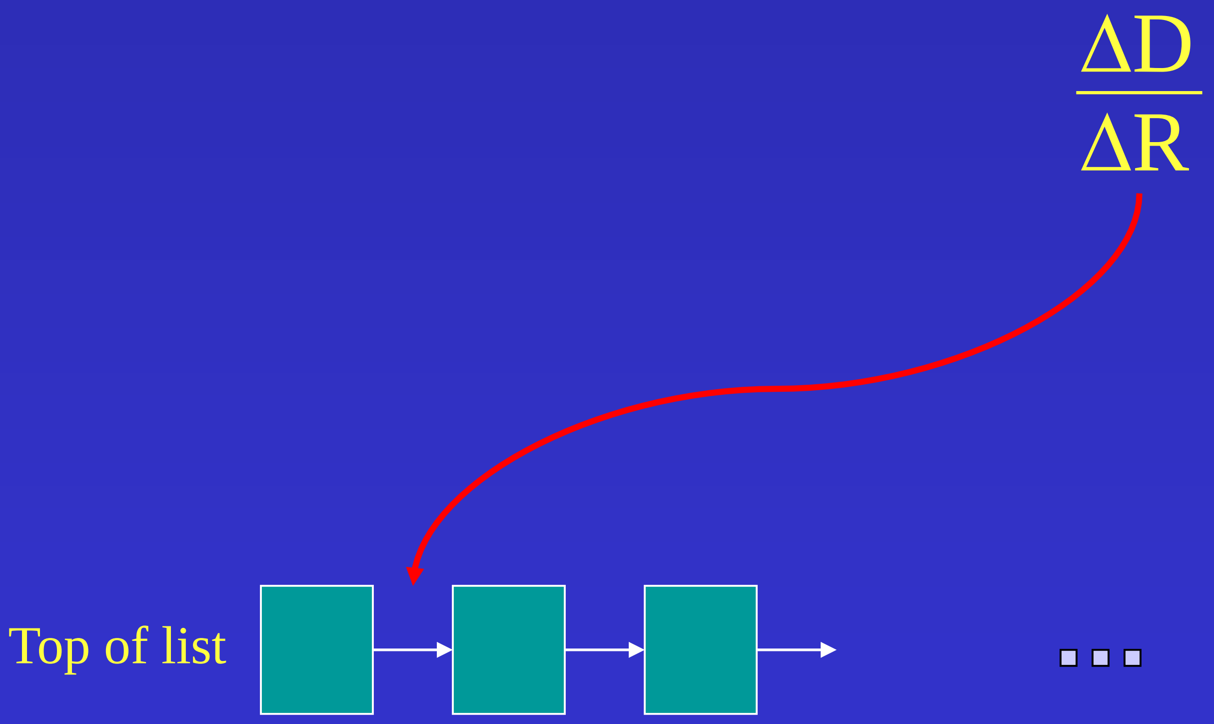
- Bits needed to describe w_i

$$\left\| E_i \right\|_2^2 = \left\| \tilde{R}_i - s_i \tilde{D}_{j_i} \right\|_2^2$$ - Collage error



Quadtree Image Partitioning

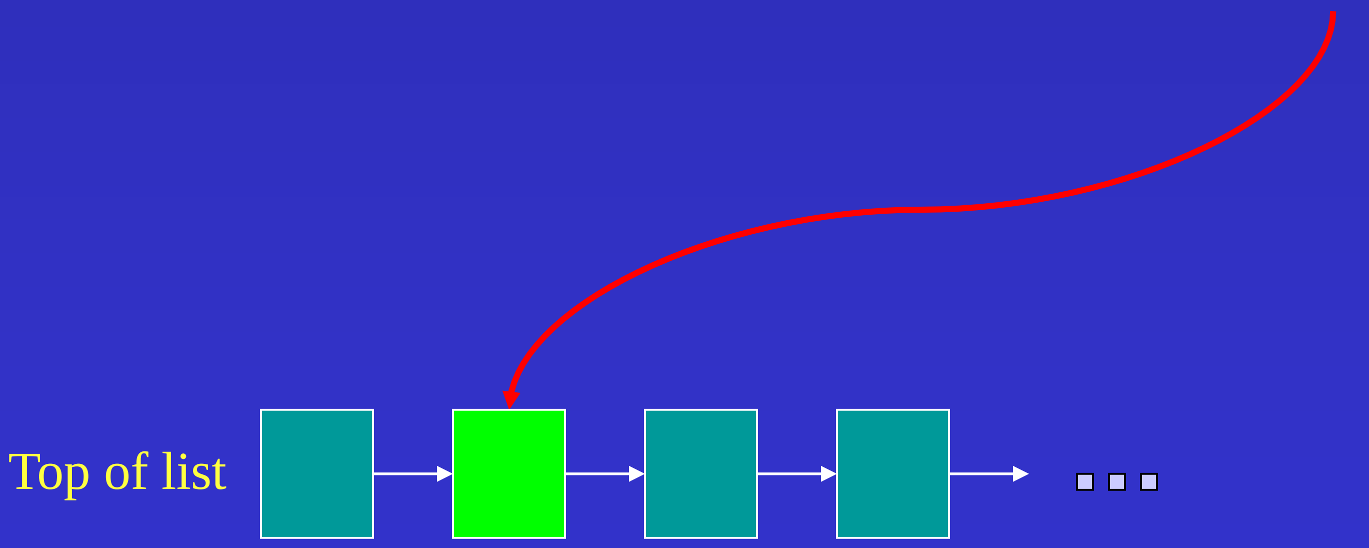
Rate-Distortion - based splitting criterion (cont'd)



Quadtree Image Partitioning

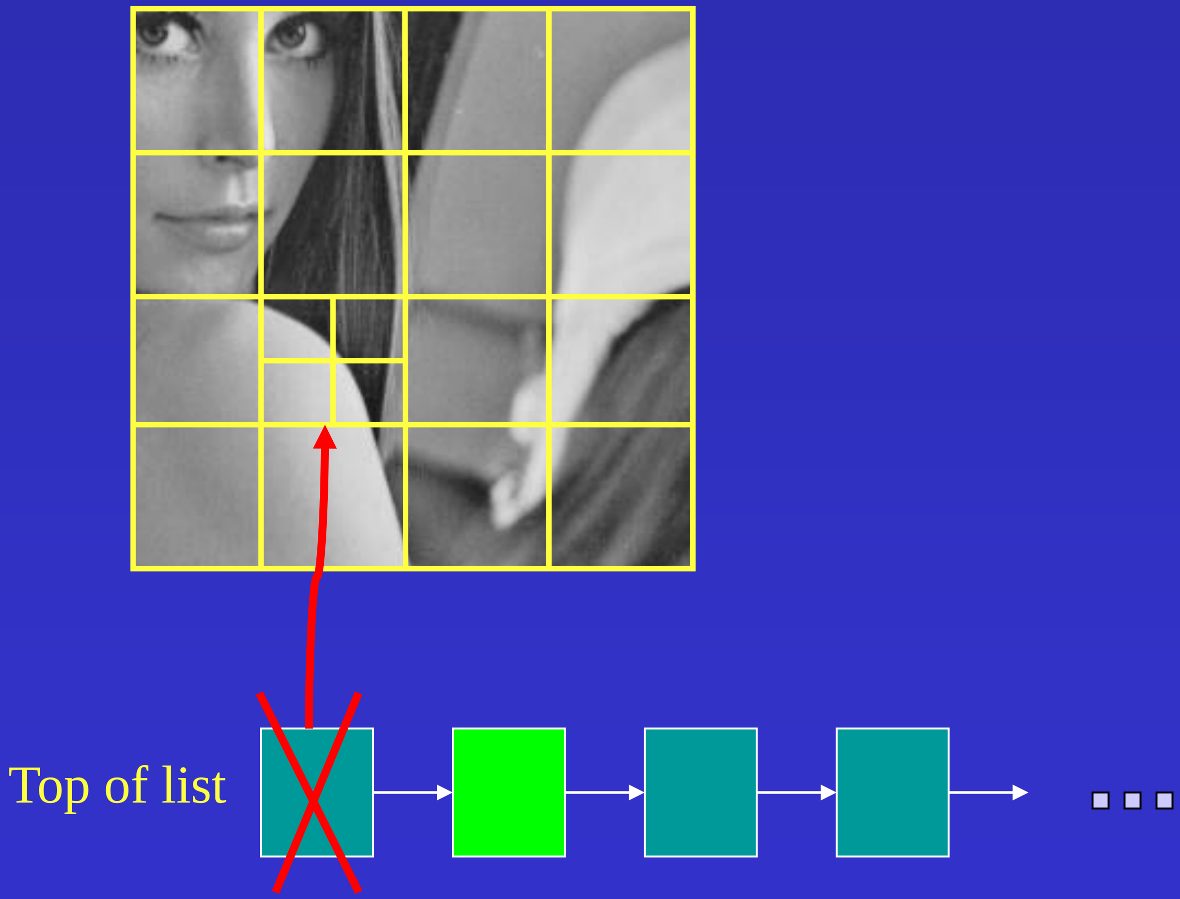
Rate-Distortion - based splitting criterion (cont'd)

$$\frac{\Delta D}{\Delta R}$$



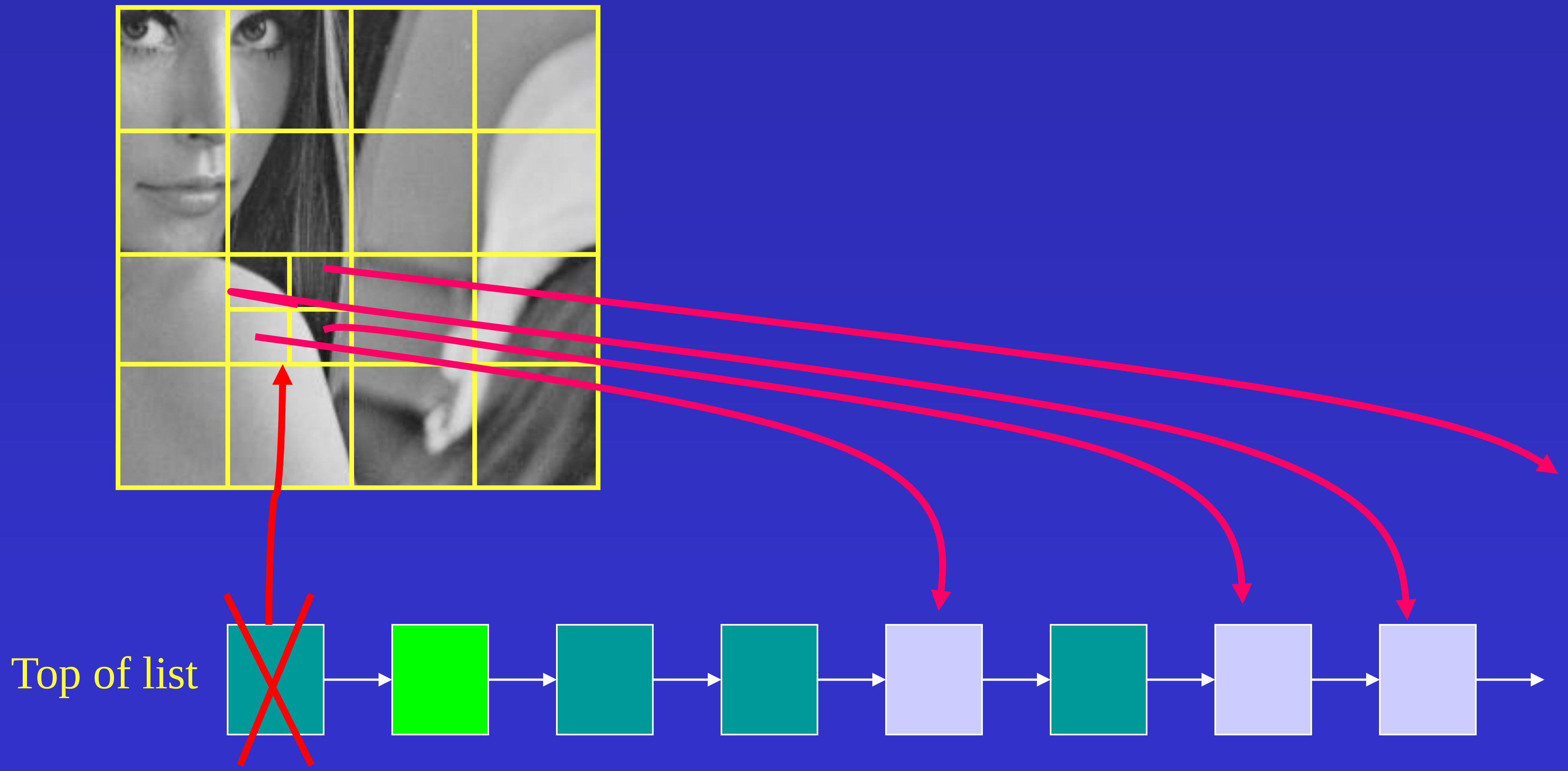
Quadtree Image Partitioning

Rate-Distortion - based splitting criterion (cont'd)



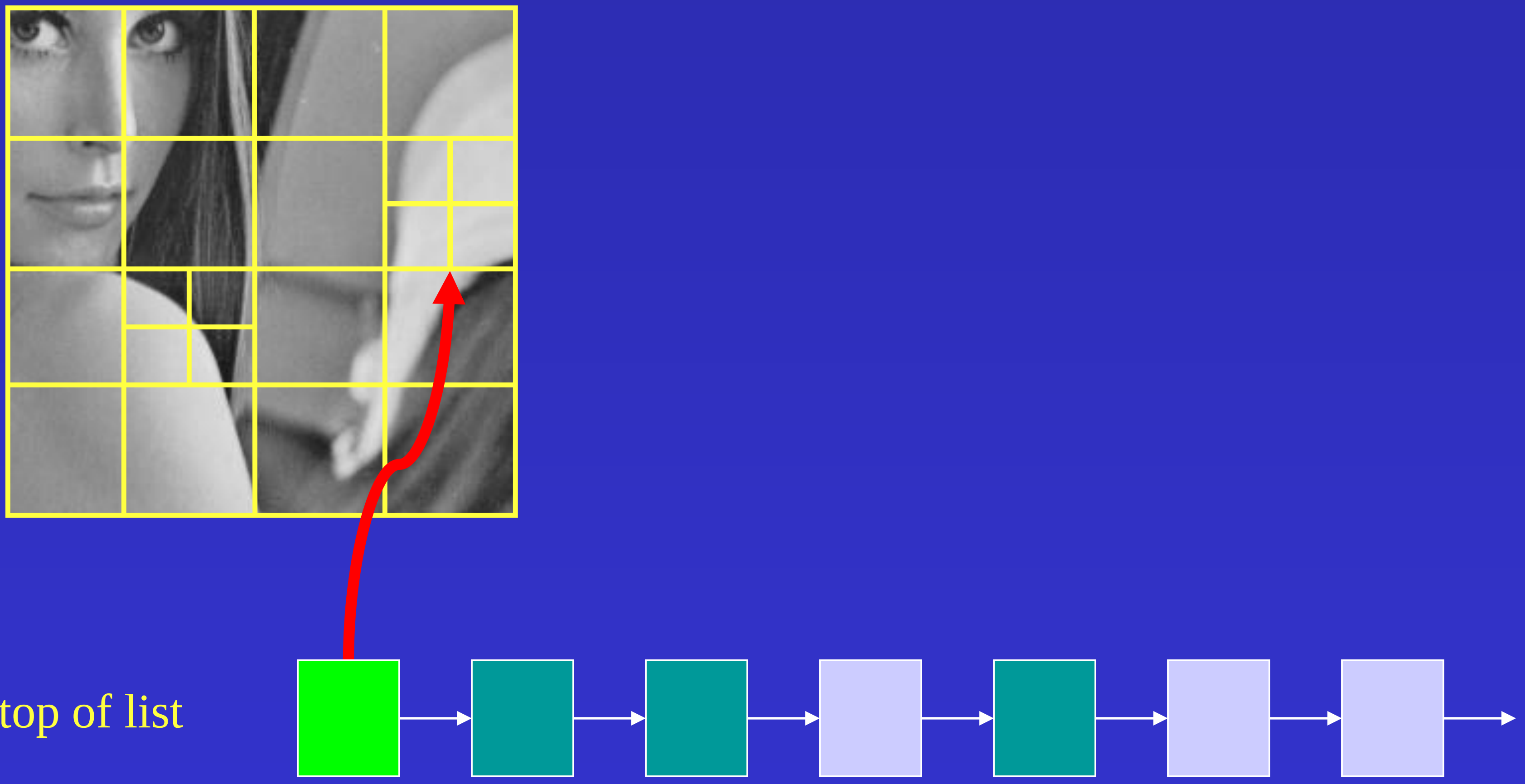
Quadtree Image Partitioning

Rate-Distortion - based splitting criterion (cont'd)



Quadtree Image Partitioning

Rate-Distortion - based splitting criterion (cont'd)



New top of list

Comparison of Splitting Criteria

Threshold - based splitting criterion :

- √ Local decision - Independent decision for each range block.
- √ A good model for determining the threshold is not available.
- √ No direct control of the bit rate.

Rate-Distortion - based splitting criterion :

- √ The whole tree structure is taken into consideration for each splitting decision.
- √ Direct control of the bit rate or the collage error.
- √ One splitting level is practically enough.
- √ Computational complexity is almost four times than threshold - based splitting criterion.

Collage-Error Computational-Complexity Splitting Criterion

For range block $\tilde{R}_i$ Gain G_{CCi} , is defined as :

$$Gcc_i = \frac{\text{Collage error}}{\text{Added complexity}}$$

Priority is given to high Gain, e.g., a block with high Collage error but with a small number of computations for reducing it.

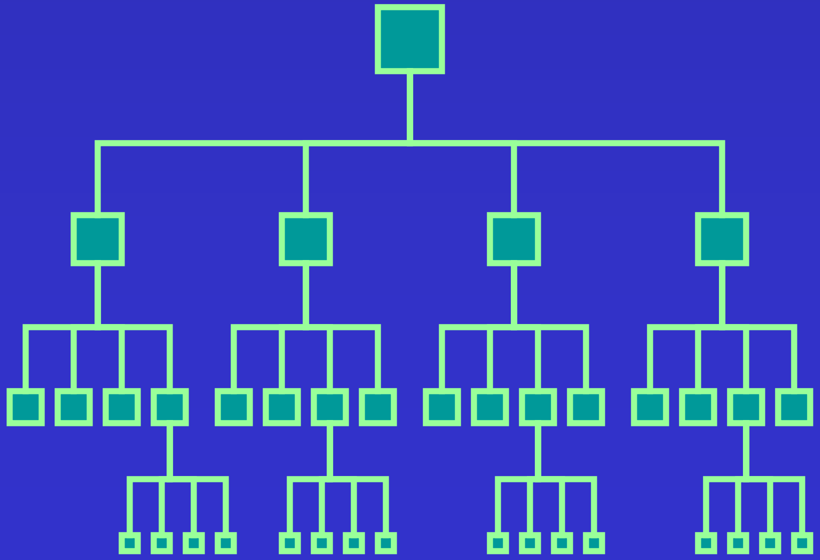
The algorithm : Keep splitting according to a descending Gain list until a designated complexity is achieved.

Adaptive Fractal Image Coding under Complexity and Rate Constraints

Using two top-down passes

First pass, find Quadtree structure using G_{CC} under Complexity constraint

Second pass, find sub-tree using G_{RD} under Rate constraint



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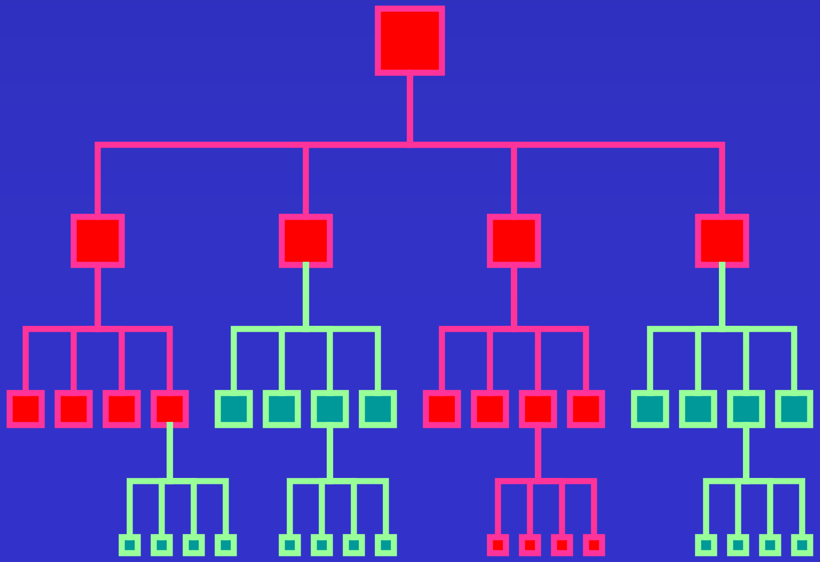

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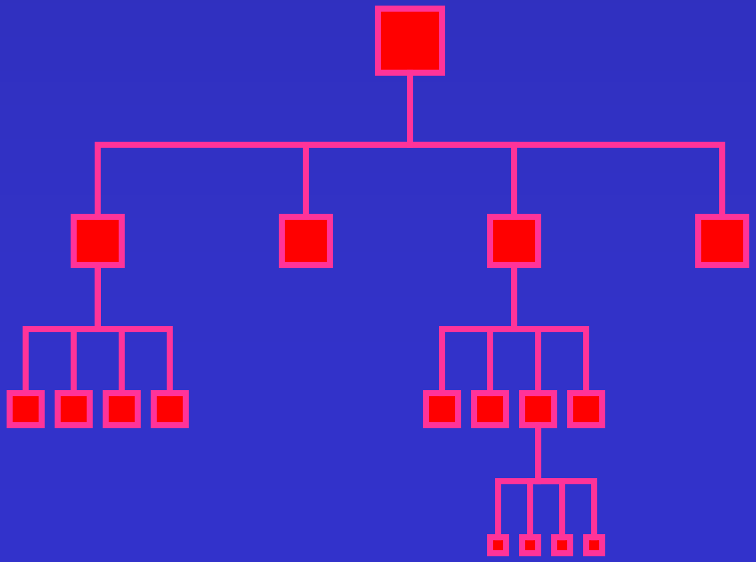
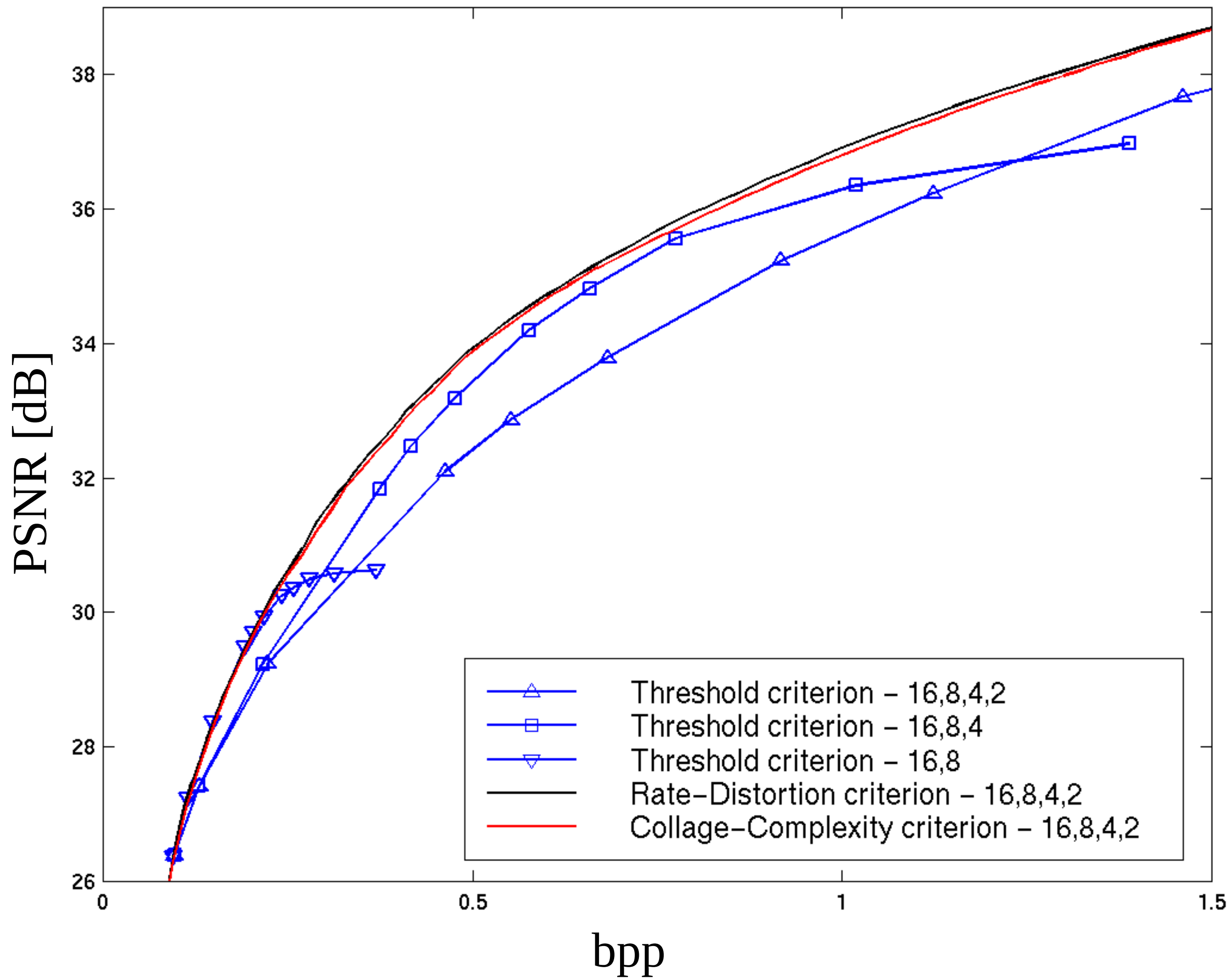


Results - “Lena” Image

PSNR vs. Rate



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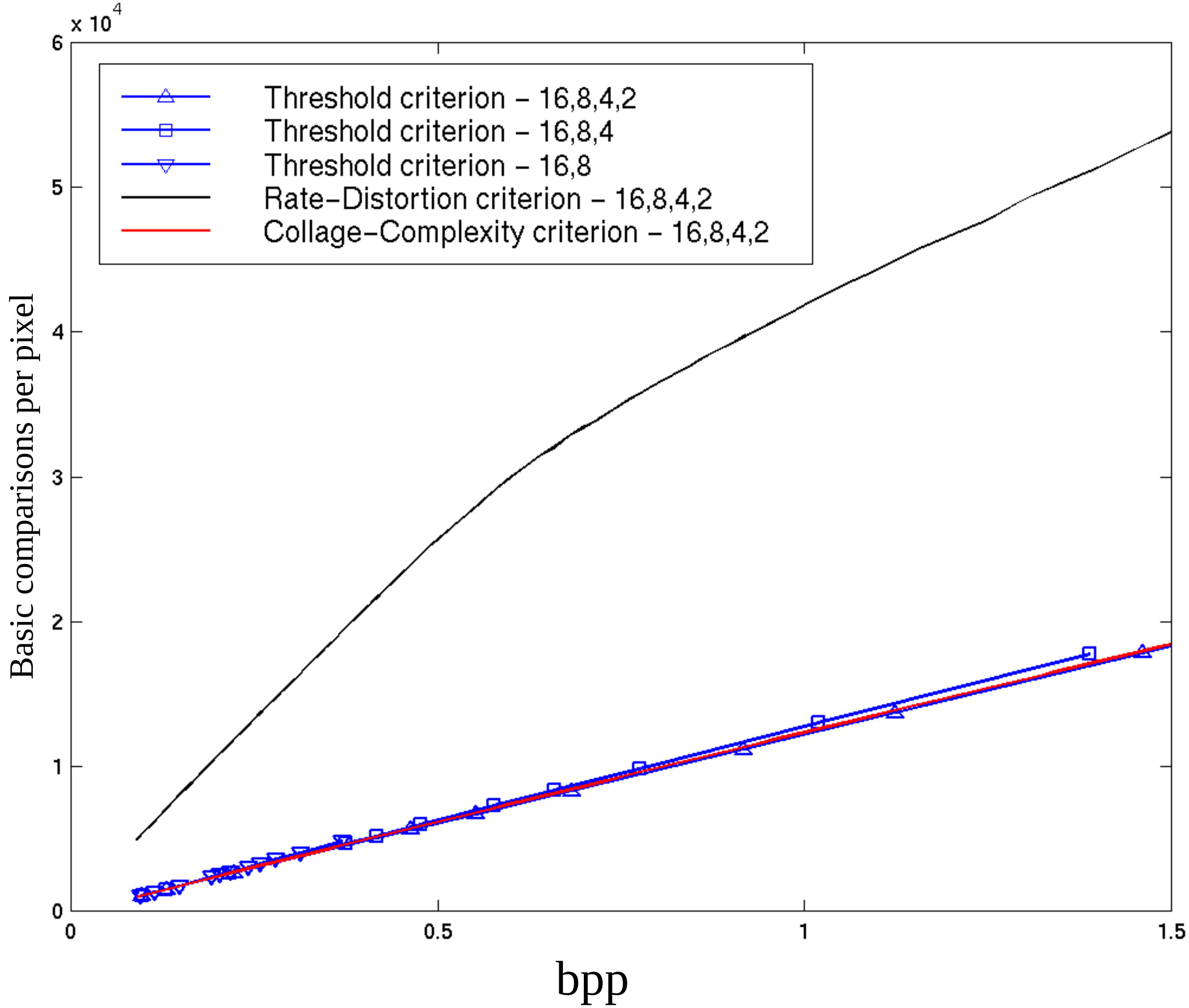


Comparison of Splitting Criteria

compression ratio ≅ 1:8

Threshold- based criterion

PSNR ≅ 35.9 [dB]

Rate-Distortion - based criterion

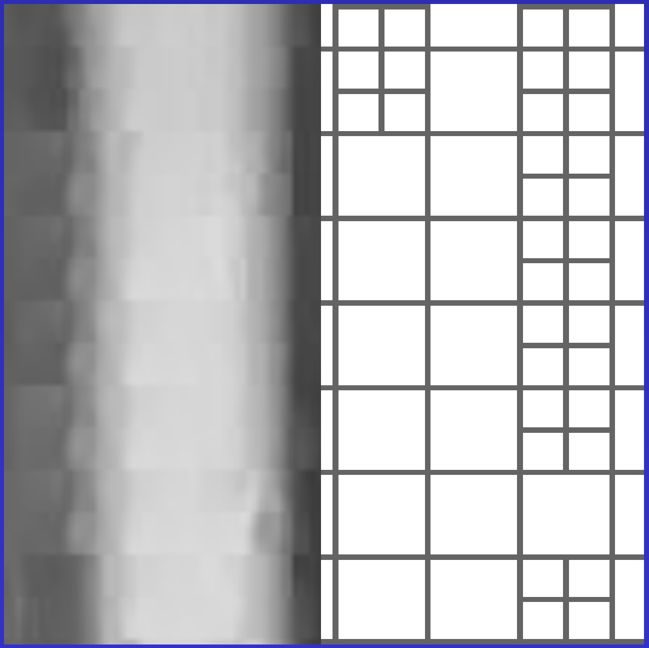
PSNR ≅ 36.9 [dB]

Range block sizes = 16x16, 8x8, 4x4, 2x2

Comparison of Splitting Criteria (cont'd)

compression ratio ≃ 1:8

Threshold- based criterion



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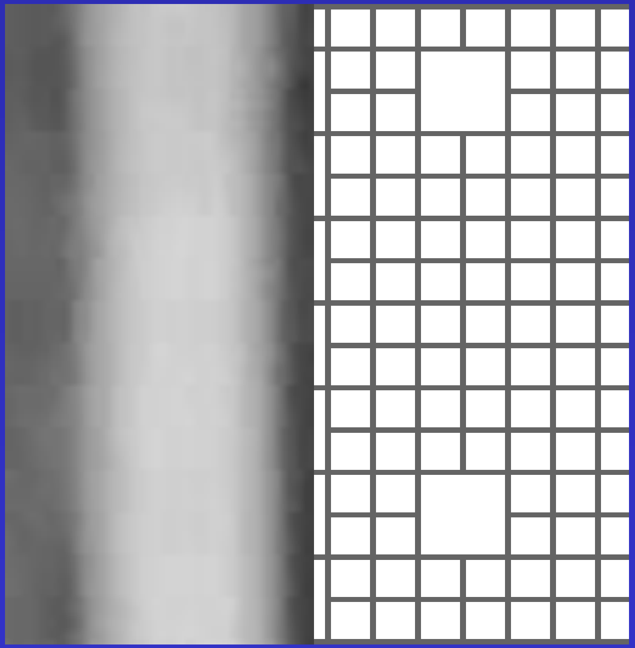
Rate-Distortion - based criterion



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Comparison of Splitting Criteria (cont'd)

compression ratio ≅ 1:8

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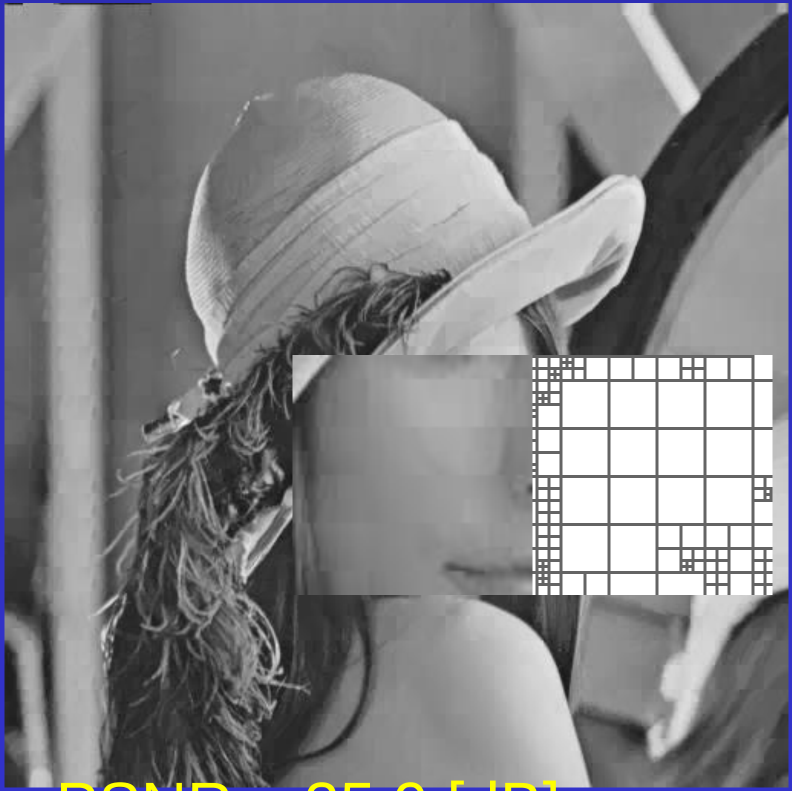

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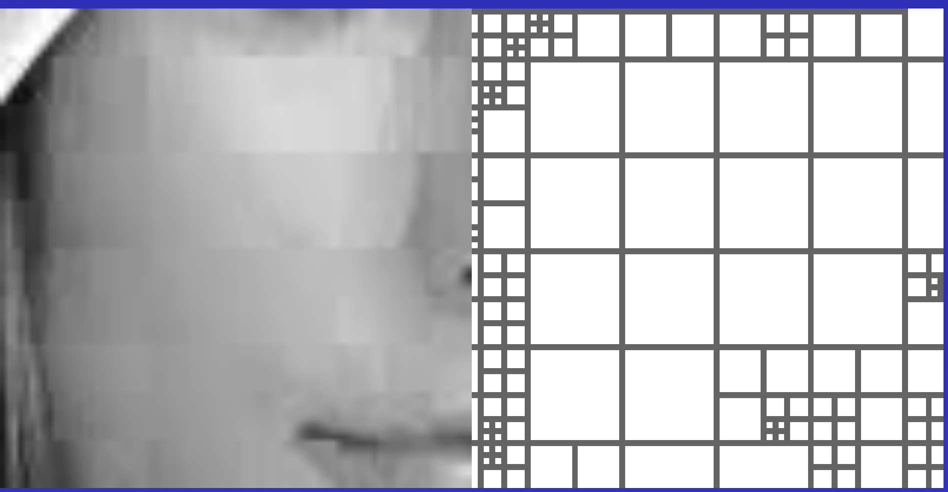

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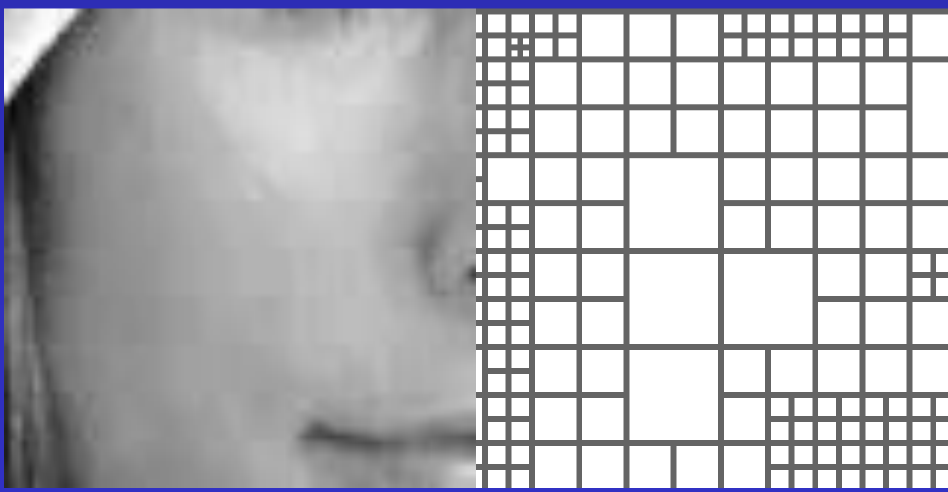
Rate-Distortion - based criterion



PSNR ≃ 36.9 [dB]

Range block sizes = 16x16, 8x8, 4x4, 2x2

Quadtree Partitioning Without Search

Motivation: Reducing complexity by searching less blocks

Coding algorithm:

a. Determine a Quadtree structue without search.

Given the Quadtree structure:

b. Search for the the best transformations.

Uses a descending order Gain list where Gain is defined using only block variances

1. Variance-Rate splitting criterion

$$G_{VRi} = \frac{\text{Size} \cdot \text{Variance}}{\text{Rate}}$$

2. ΔVar-Rate splitting criterion

$$G_{Vi} = \frac{\text{Size} \cdot \text{Variance Decrease}}{\text{Rate}}$$

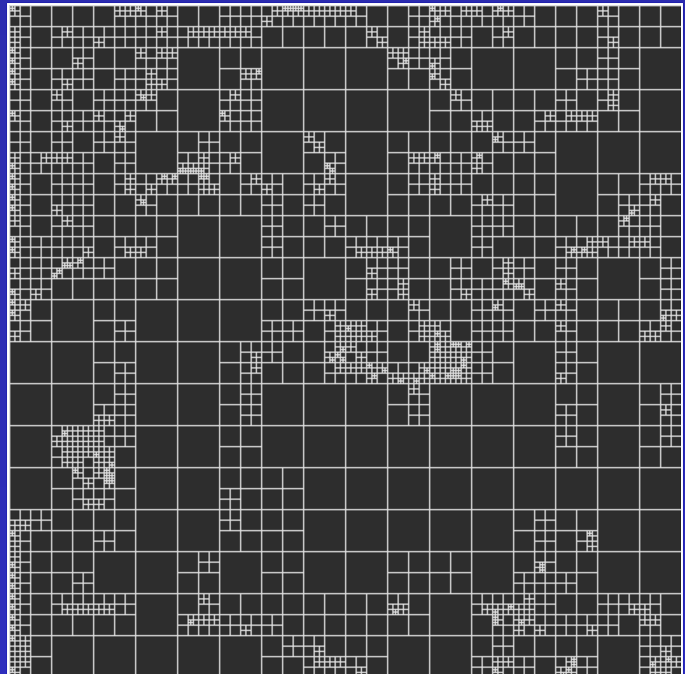
Results - Summary

"Peppers" Image - 0.3bpp

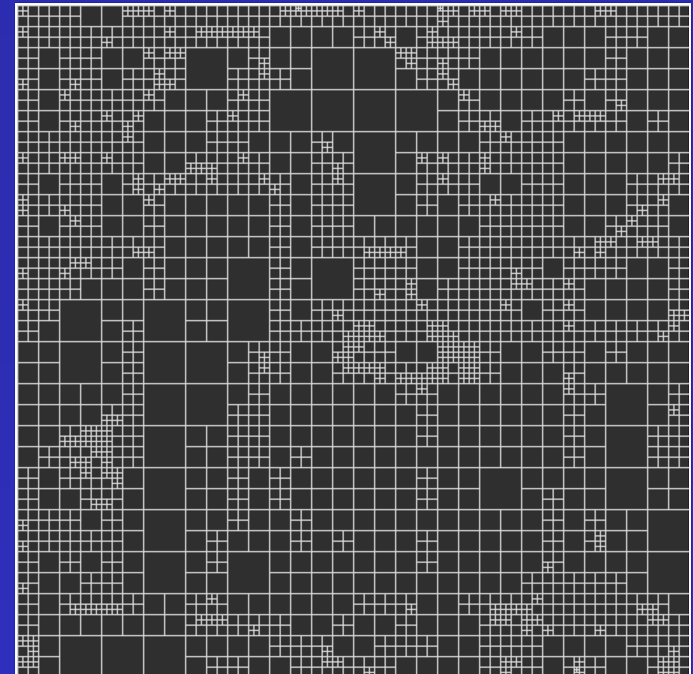
Range Blocks used 32,16,8,4 and 2

Split Criterion	Collage Error [dB]	Reconstruction Error [dB]	Complexity [%]
Rate-Distortion	30.74	30.25	100
Collage error-Computational complexity	30.62	30.15	25
Variance-Rate	29.73	29.21	18.8
Threshold	29.14	28.02	24
Rate-Distortion & Segmentation	28.48	27.81	5
ΔVar-Rate	28.31	26. 83	19

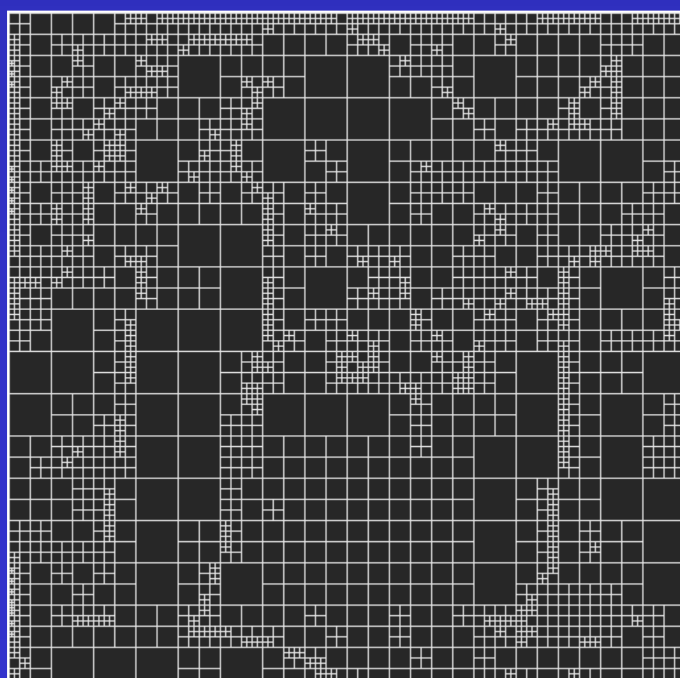
Threshold



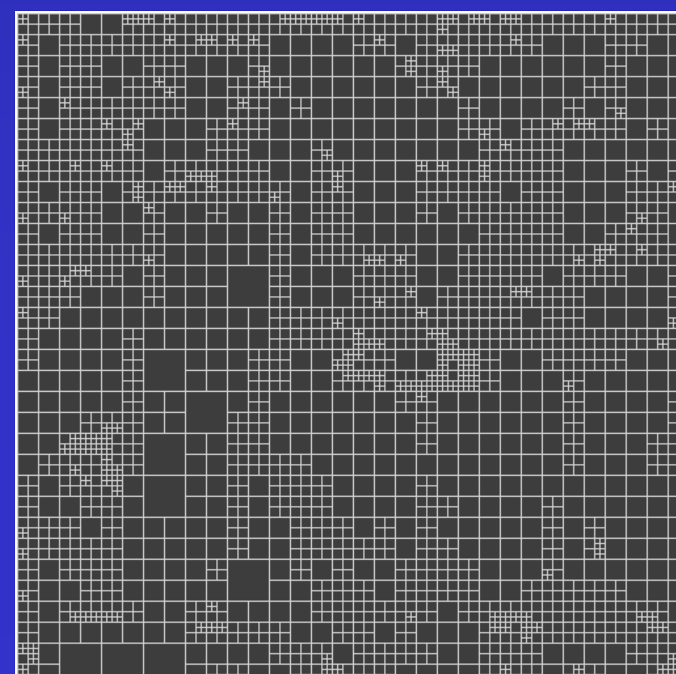
Rate-
Distortion



Variance-
Rate

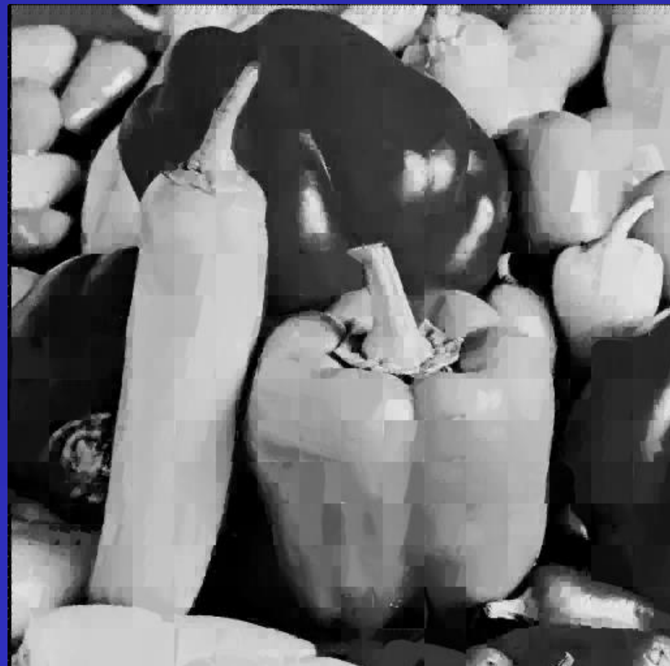


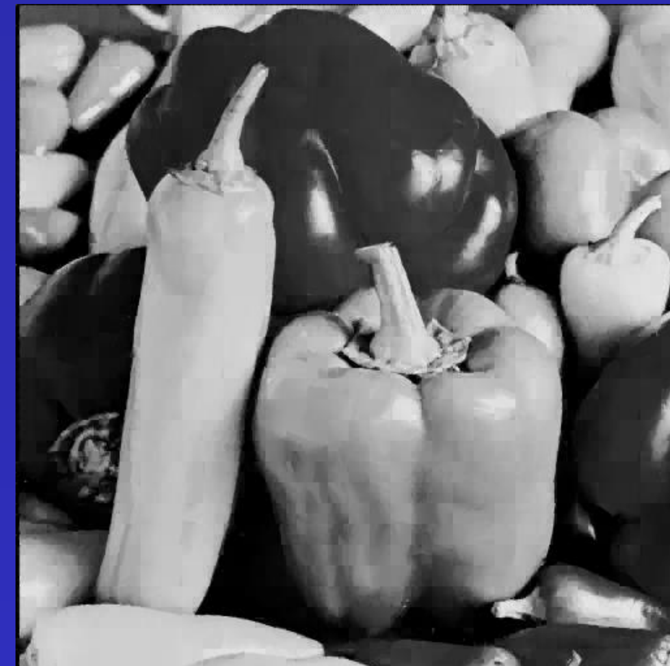
Collage
error-
Complexity



Threshold

28.02dB
0.3bpp





Rate-
Distortion

30.25dB
0.3bpp

Variance-
Rate

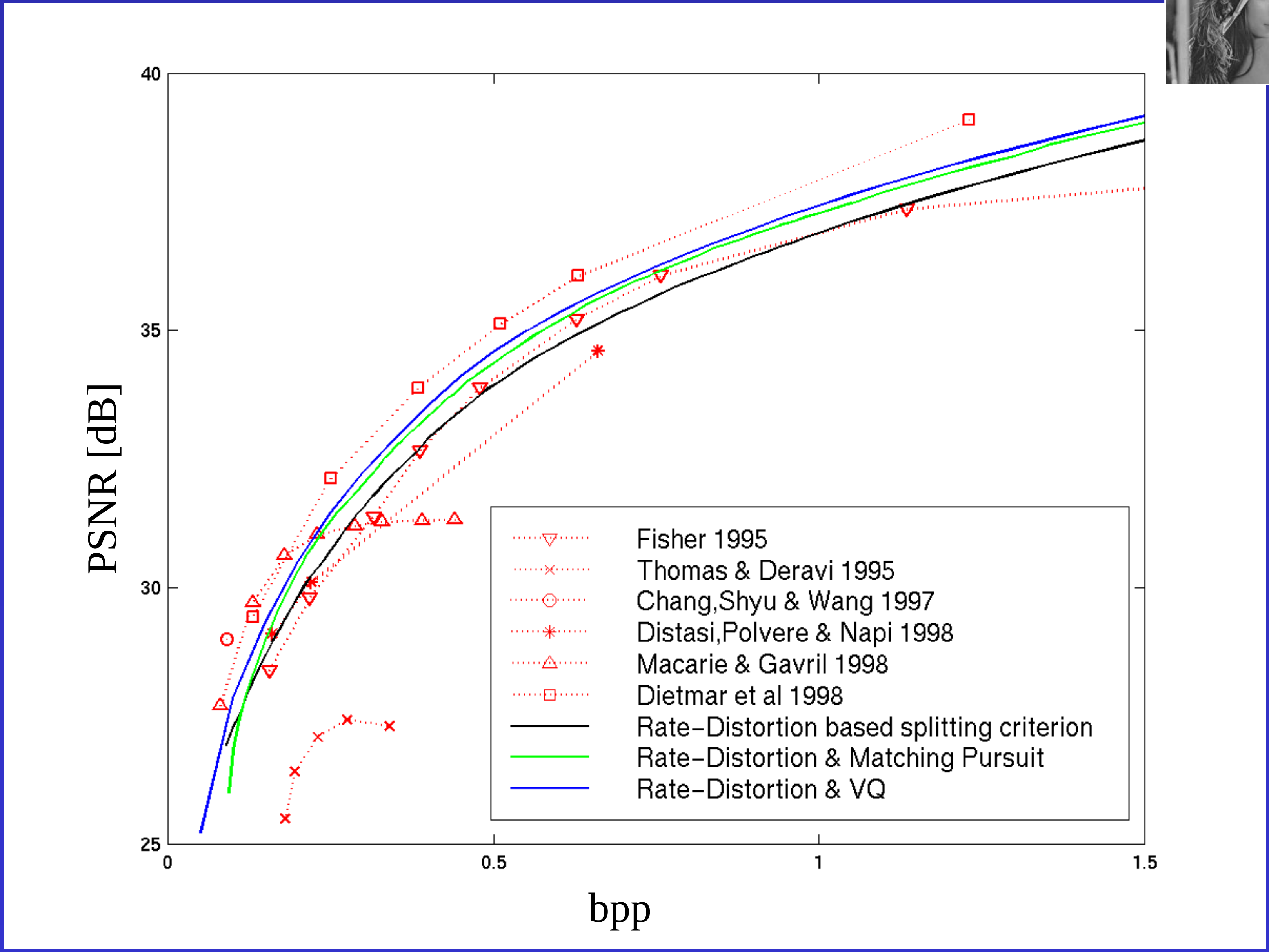
29.21dB
0.3bpp

Collage
error-
Complexity

30.15dB

0.3bpp

Comparison to related works



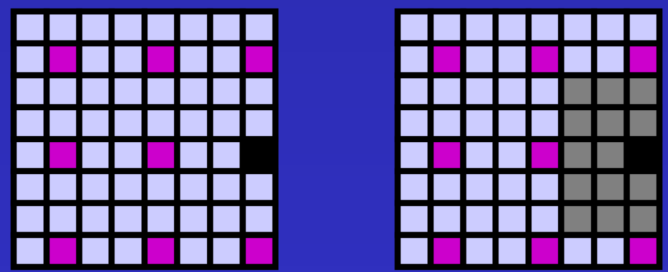
"Fast Search"

\# Range Blocks $\approx$ # Domain Blocks $\approx$ N

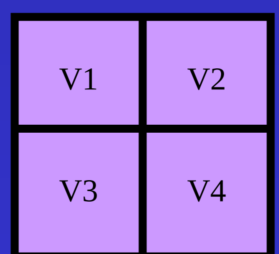
"Full Search" - check all domain blocks for each range block.

"Full Search" is of complexity $O(N^2)$

Decimated search



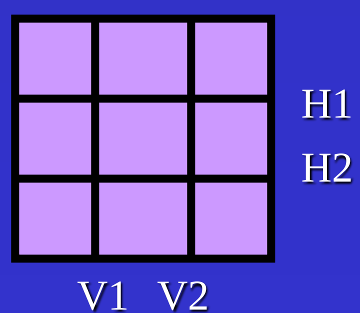
Classification according to variance order



$$V_1 \geq V_2 \geq V_3 \geq V_4$$

$$V_4 \geq V_3 \geq V_2 \geq V_1$$

Classification according to zero crossing



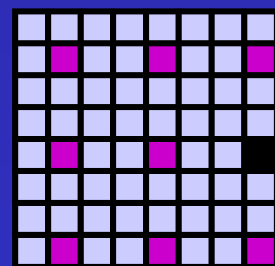
H1	H2	V1	V2

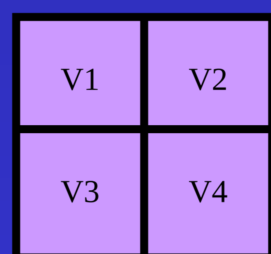
Typical vector

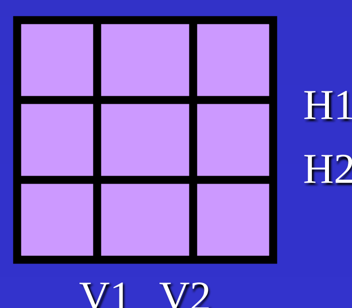
"Fast Search" (cont'd)

"Lena" Image - 0.3bpp Range blocks 16,8 & 4

	PSNR Reduction	Complexity "effort" [%]
Decimated search	-1.66 dB	4.6 %
Classification according to variance order	-1 dB	4.5 %
Classification according to zero crossing	-0.94 dB	16 %







Range Block Matching with Multi Domain Blocks

Problem definition: Find the Kth order linear combination of domain blocks that minimize the MSE

$$\arg \min_{\vec{S},\, DOMAINS} \left\| \tilde{R} - s_1 \tilde{D}_1 - s_2 \tilde{D}_2 - \ldots - s_K \tilde{D}_K \right\|_2^2$$

where: $\vec{S} = \left(s_1 \; s_2 \; s_3 \; \ldots \; s_K \right)^t \quad s_j \in \Re$

Solution: Given a set of K domain blocks, calculate :

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ \\ s_K \end{pmatrix} = \begin{pmatrix} \|\tilde{D}_1\|^2 & \langle \tilde{D}_1, \tilde{D}_2 \rangle & & \cdots & \langle \tilde{D}_1, \tilde{D}_K \rangle \\ \langle \tilde{D}_1, \tilde{D}_2 \rangle & \|\tilde{D}_2\|^2 & & & \vdots \\ & & \ddots & & \\ \vdots & & & \|\tilde{D}_{K-1}\|^2 & \langle \tilde{D}_{K-1}, \tilde{D}_K \rangle \\ \langle \tilde{D}_1, \tilde{D}_K \rangle & \cdots & & \langle \tilde{D}_{K-1}, \tilde{D}_K \rangle & \|\tilde{D}_K\|^2 \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{R}, \tilde{D}_1 \rangle \\ \langle \tilde{R}, \tilde{D}_2 \rangle \\ \vdots \\ \\ \langle \tilde{R}, \tilde{D}_K \rangle \end{pmatrix}$$

For a small K, solution is of $O(N^3)$ Complexity or $O(N^2)$ if using $O(N^2)$ memory units (not practical).

Sub-Space Orthogonaliztion [Øien 1991]

Define K-1 order orthonormal basis : $\{b_1, b_2, \dots b_{K-1}\}$

Find : $\arg\min_{\vec{s},j} \left\| R - s_1 b_1 - s_2 b_2 - \dots - s_{K-1} b_{K-1} - s_K D_j \right\|_2^2$

Range and Domain blocks can be orthogonalized, in advance, to the space spanned by $\{b_1, b_2, \dots b_{K-1}\}$.

Solution:

$$\begin{pmatrix} 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ & 0 & \ddots & \ddots & \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & & 0 & \left\|\tilde{D}_j\right\|^2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{K-1} \\ s_K \end{pmatrix} = \begin{pmatrix} \langle \tilde{R}, b_1 \rangle \\ \langle \tilde{R}, b_2 \rangle \\ \vdots \\ \langle \tilde{R}, b_{K-1} \rangle \\ \langle \tilde{R}, \tilde{D}_j \rangle \end{pmatrix}$$

$$s_K = \frac{\langle \tilde{R}, \tilde{D}_j \rangle}{\left\|\tilde{D}_j\right\|^2}, \quad s_m = \langle \tilde{R}, b_m \rangle, \; 1 \le m \le K-1$$

Matching Pursuit [Mallat 1993]

Decomposing a signal into linear expansion using waveforms selected from a redundant dictionary of functions.

1. Denote the source signal as a residual signal.

2. Find in a dictionary (of size M) the function with the highest correlation factor with the residual signal.

3. Find the new residual signal using projection on the selected function.

Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit

Problem definition: Find best 2nd order linear combination of domain blocks to minimize collage error.

Optimal solution: Given a pair of domain blocks, calculate :

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \frac{\begin{pmatrix} \left\|\tilde{D}_2\right\|_2^2 & -\left\langle \tilde{D}_1, \tilde{D}_2 \right\rangle \\ -\left\langle \tilde{D}_1, \tilde{D}_2 \right\rangle & \left\|\tilde{D}_1\right\|_2^2 \end{pmatrix} \begin{pmatrix} \left\langle \tilde{R}, \tilde{D}_1 \right\rangle \\ \left\langle \tilde{R}, \tilde{D}_2 \right\rangle \end{pmatrix}}{\left\|\tilde{D}_1\right\|_2^2 \left\|\tilde{D}_2\right\|_2^2 - \left\langle \tilde{D}_1, \tilde{D}_2 \right\rangle^2}$$

Optimal solution is of O(N^3) Complexity

or O(N^2) if using O(N^2) memory units (not practical).

Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

For each range block,

find S_1 and D_1 that minimize $\left\| \tilde{R} - S_1 \tilde{D}_1 \right\|_2^2$

find S_2 and D_2 that minimize

MP1

$\left\| \tilde{R} - S_1 \tilde{D}_1 - S_2 \tilde{D}_2 \right\|_2^2$

Given S_1 and D_1

MP2

$\left\| \tilde{R} - S_1 \tilde{D}_1 - S_2 \left(\tilde{D}_2 - S_1 \tilde{D}_1 \right) \right\|_2^2$

Given S_1 and D_1

MP3

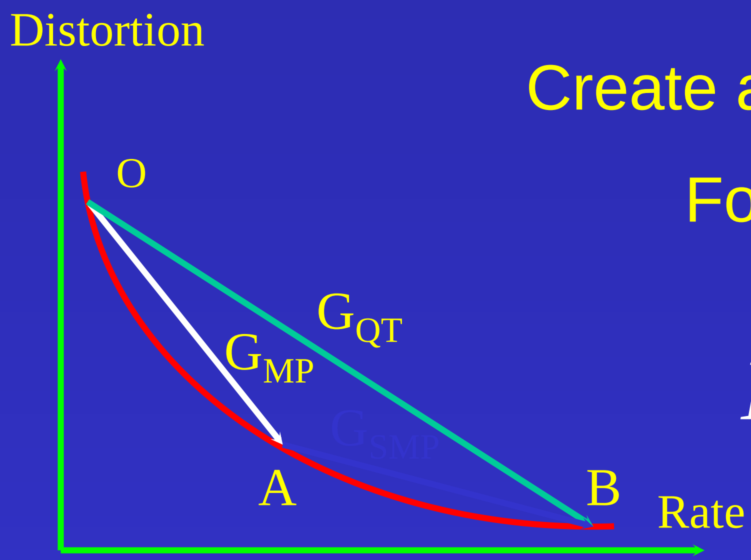
$\left\| \tilde{R} - S_1 \tilde{D}_1 - S_2 \tilde{D}_2 \right\|_2^2$

Given D_1

Complexity is two times "Full Search" - $O(N^2)$

Only O(N) memory units are required.

Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

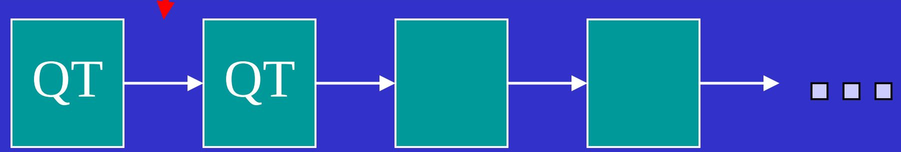


Create a descending order Gain list.

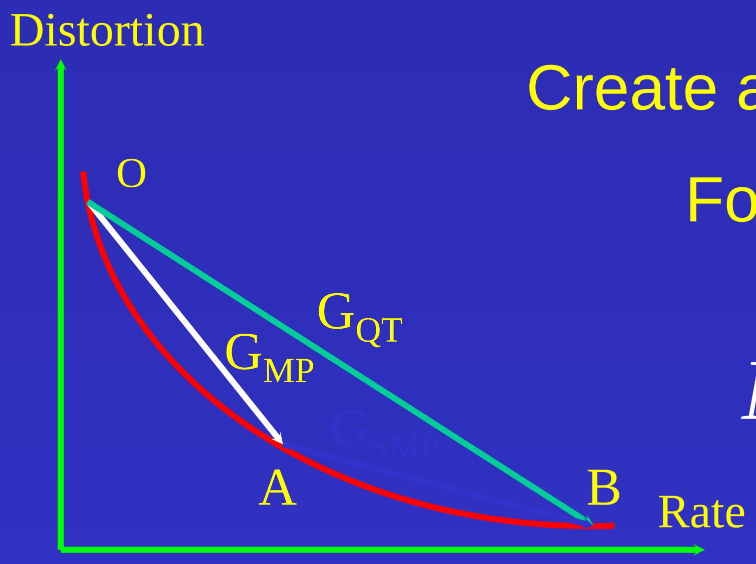
For each range block use:

$$MAX\left(G_{QT}, G_{MP} \right)$$

Top of list



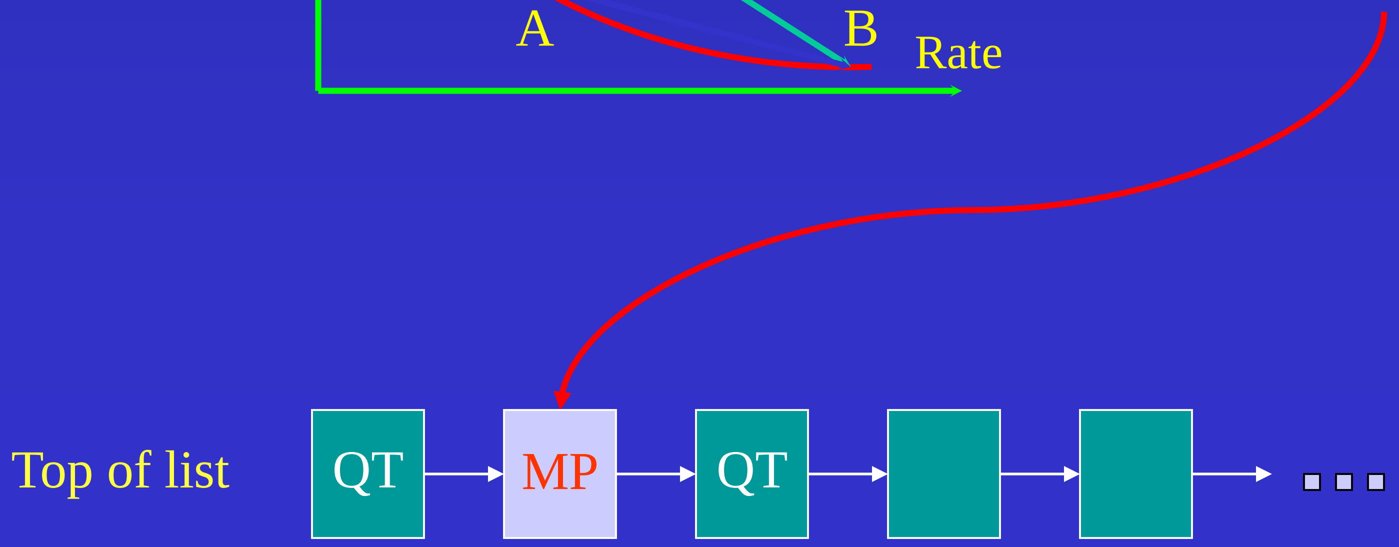
Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)



Create a descending order Gain list.

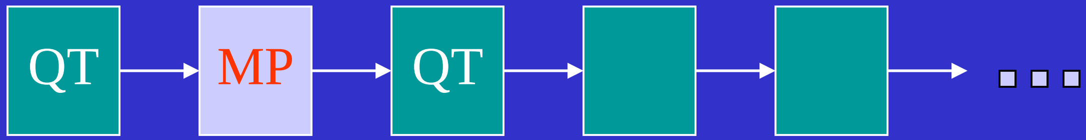
For each range block use:

$$MAX\left(G_{QT}, G_{MP}\right)$$



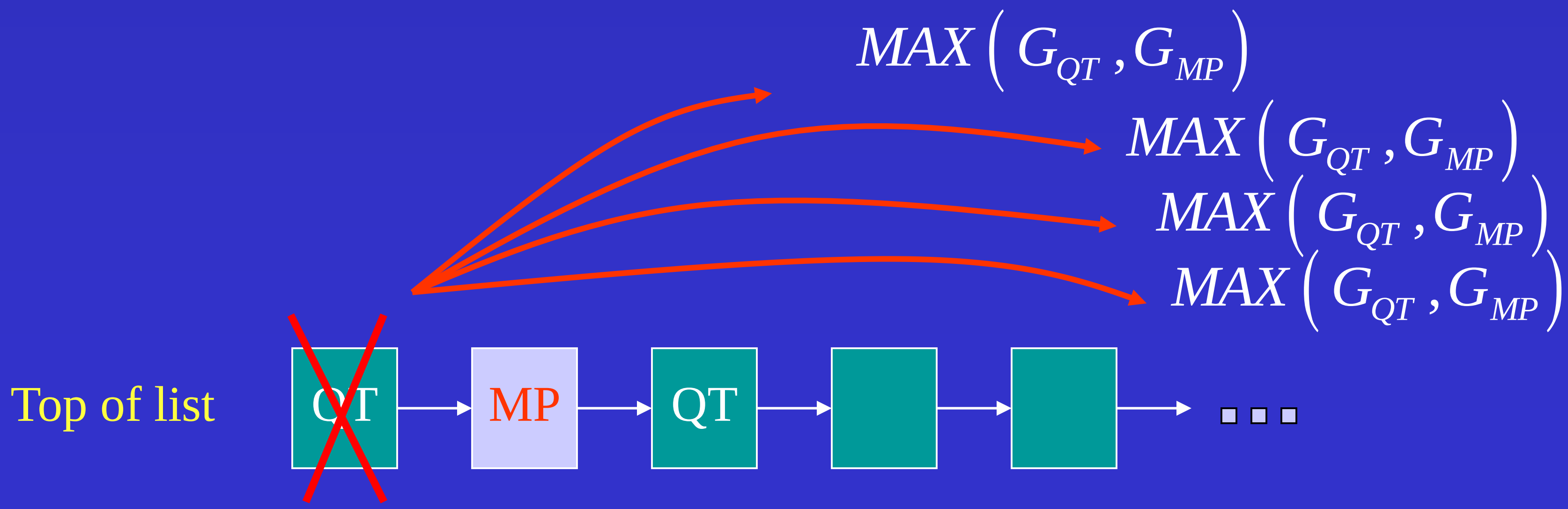
Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

Top of list

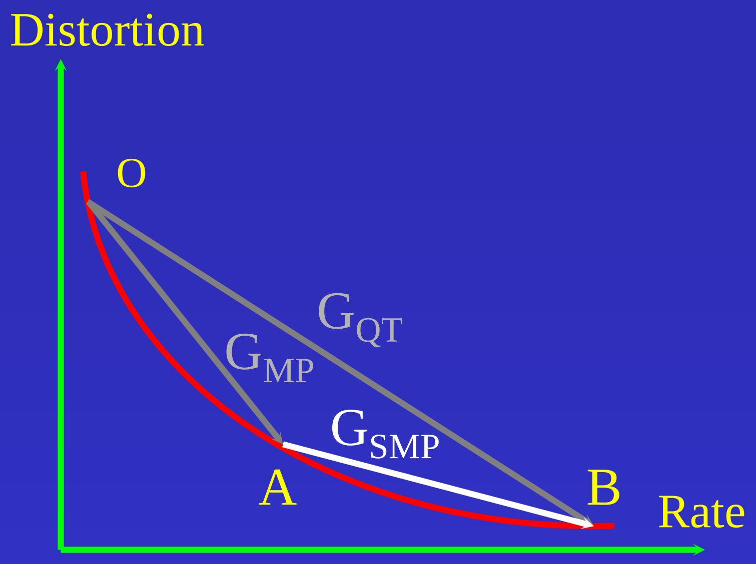


Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

Start splitting according to the ordered Gain list

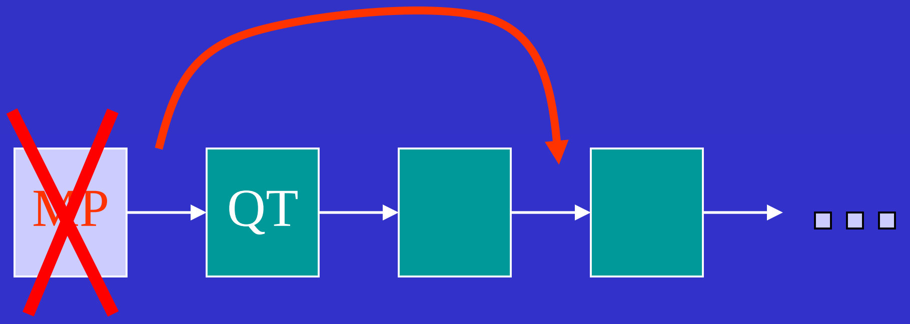


Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

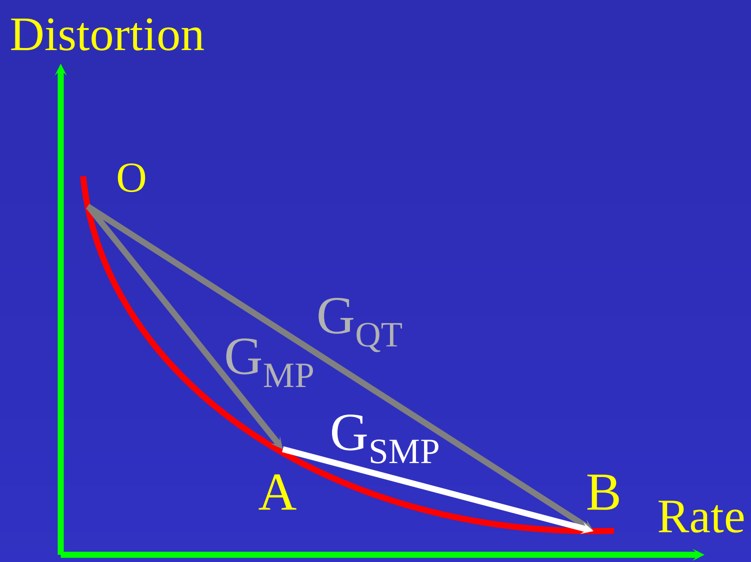


If a matching pursuit labeled (MP) block is at the top of the list place G_{SMP} in the proper location

Top of list



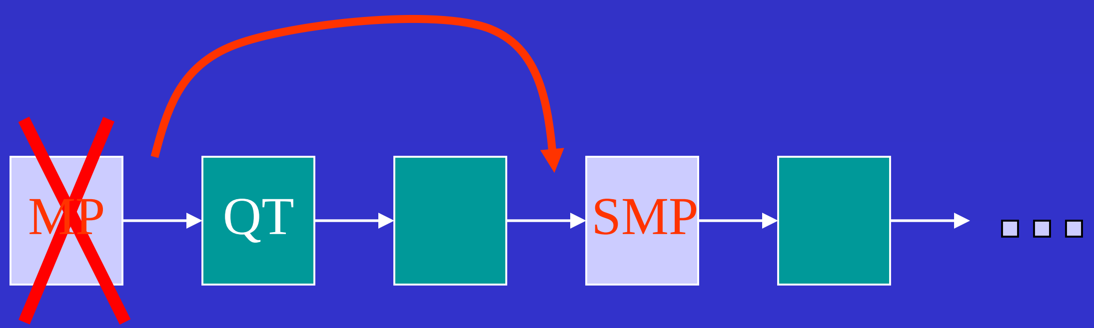
Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)



If a matching pursuit labeled (MP) block is at the top of the list place G_{SMP} in the proper location

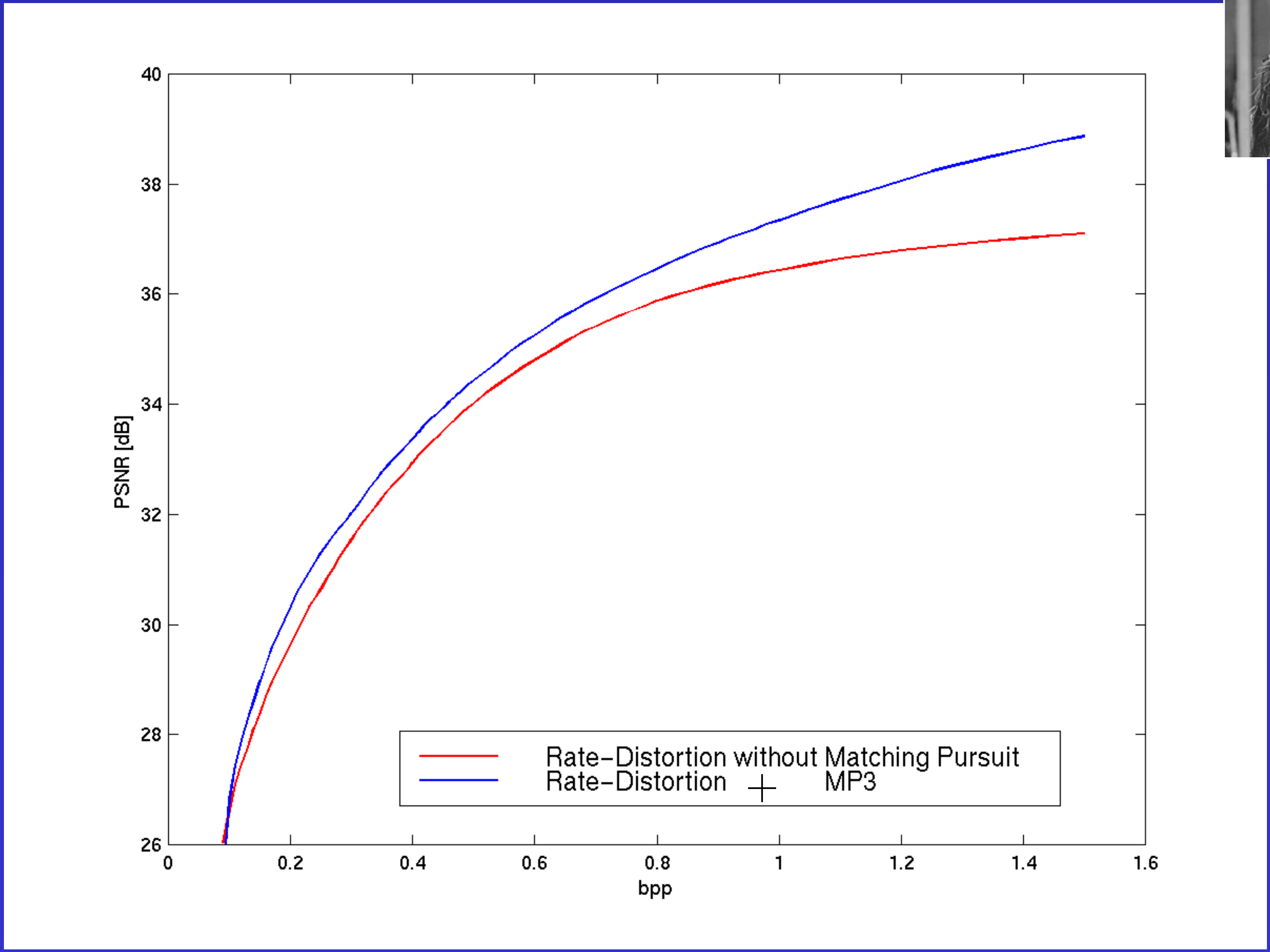
A labeled MP block is split only if G_{SMP} reaches the top of the list !

Top of list



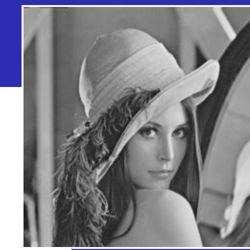
Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

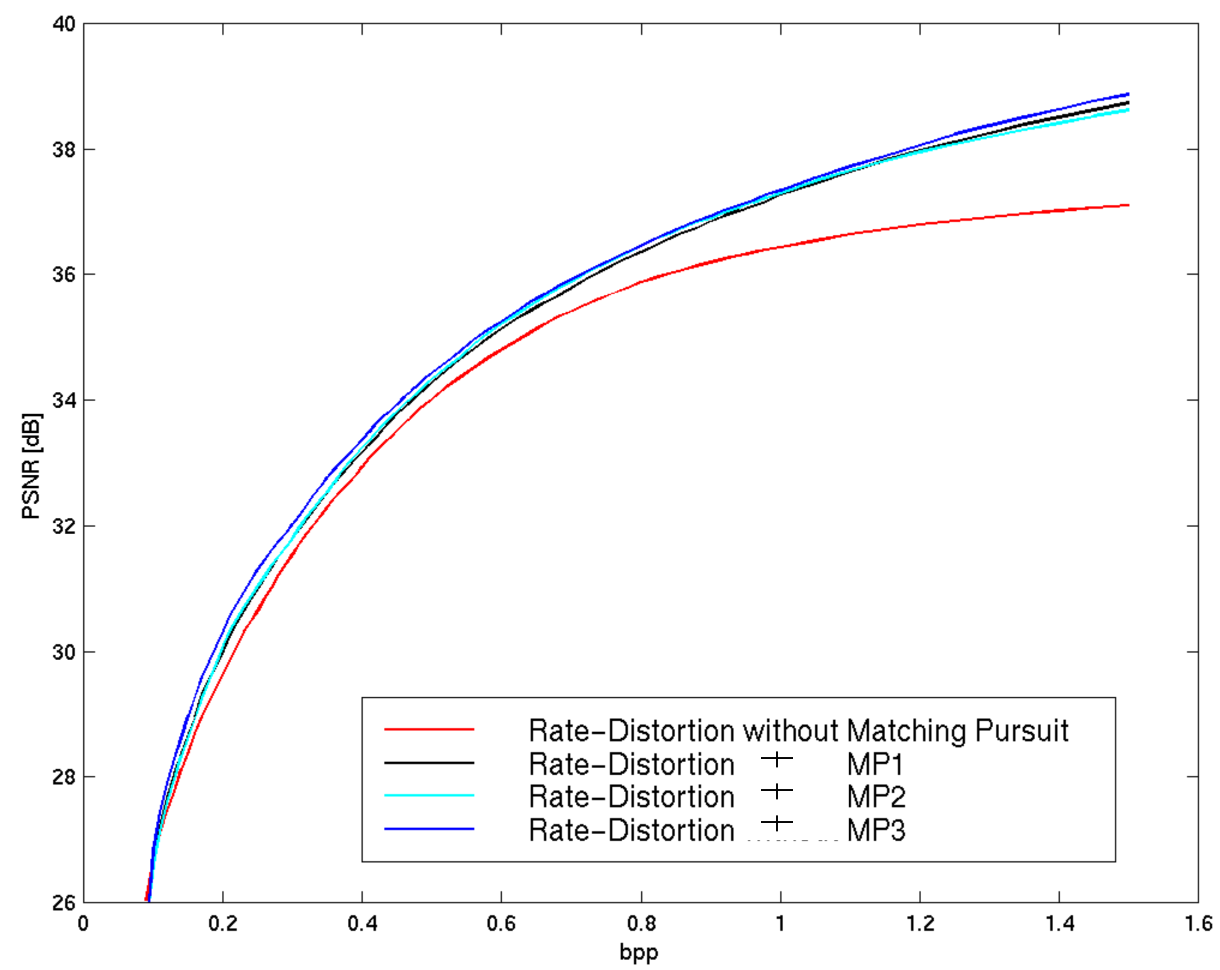
"Lena" Image , Size of range blocks : 16,8 & 4



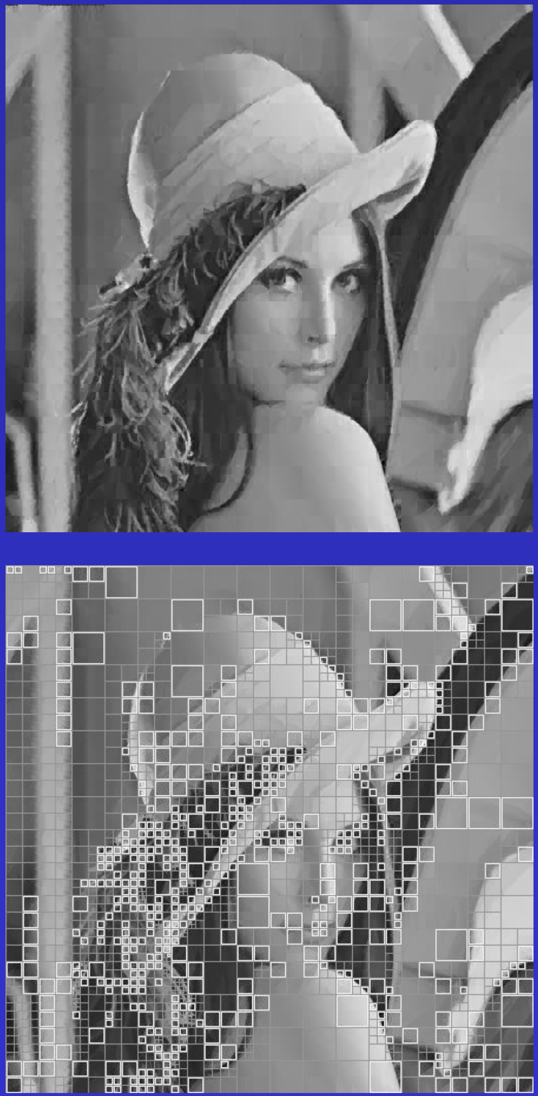
Rate-Distortion - based Splitting Criterion Combined with Matching Pursuit (cont'd)

"Lena" Image , Size of range blocks : 16,8 & 4



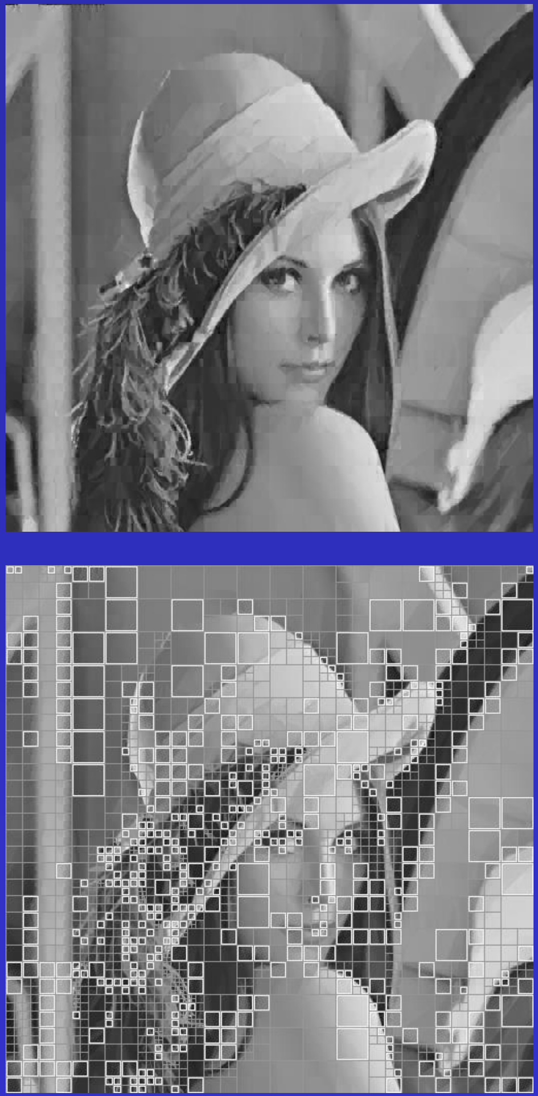


Comparison of Results for Matching Pursuit



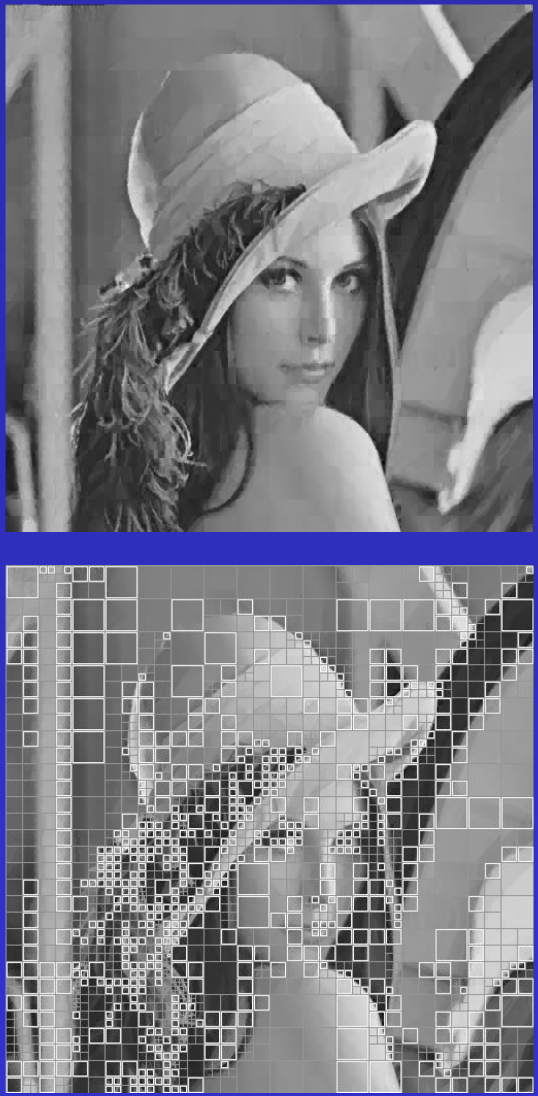
0.3 bpp PSNR=31.67dB

MP1



0.3 bpp PSNR=31.76dB

MP2



0.3 bpp PSNR=31.48dB

MP3

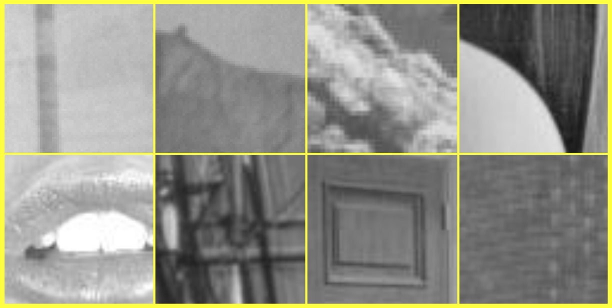
Hybrid Fractal - VQ Coding

Use external code-book to enlarge the domain pool

- "Self" Code-Book
- Signal Dependent

Domain Pool

- Defined Code-Book
- Signal independent

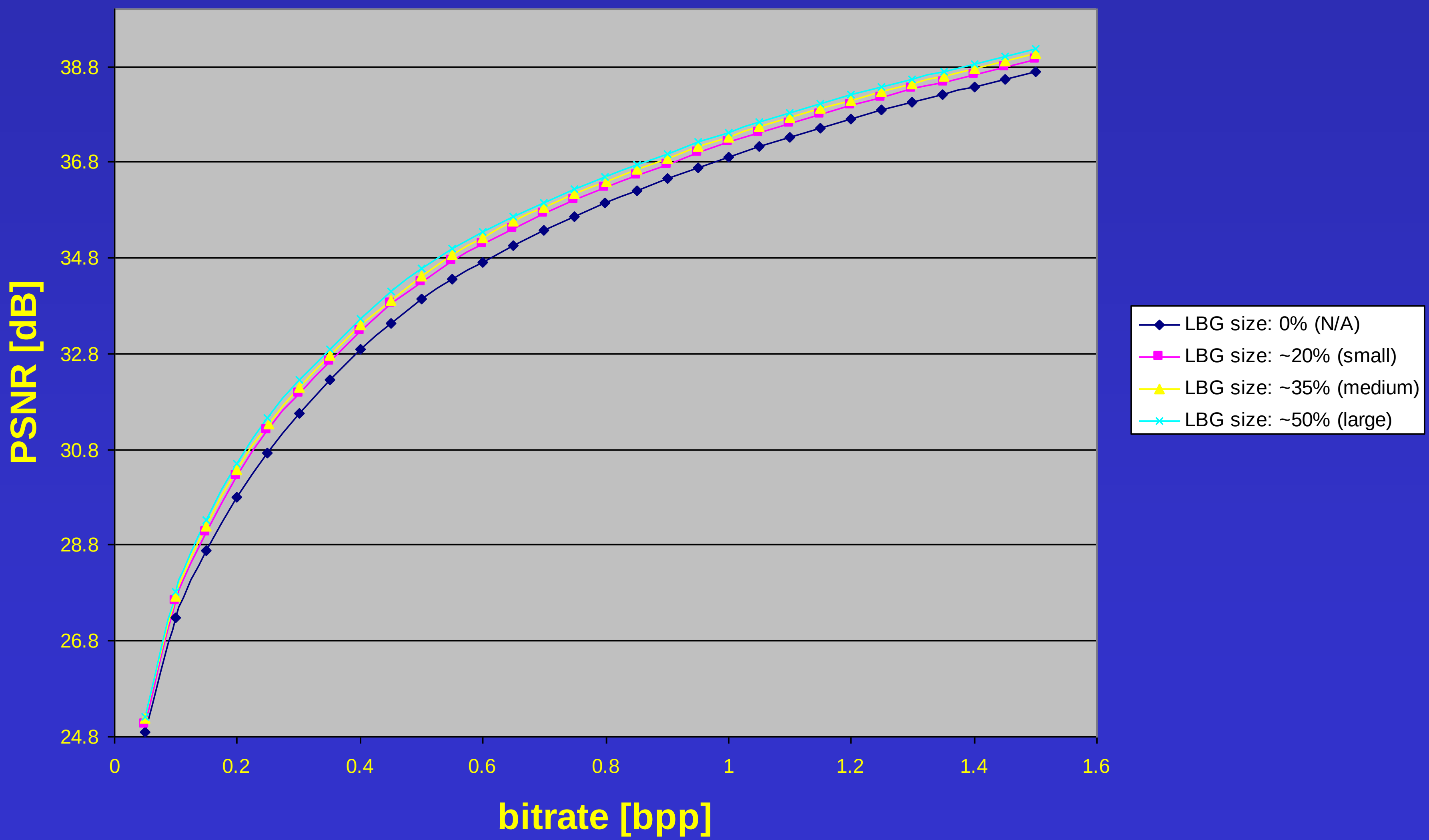


VQ

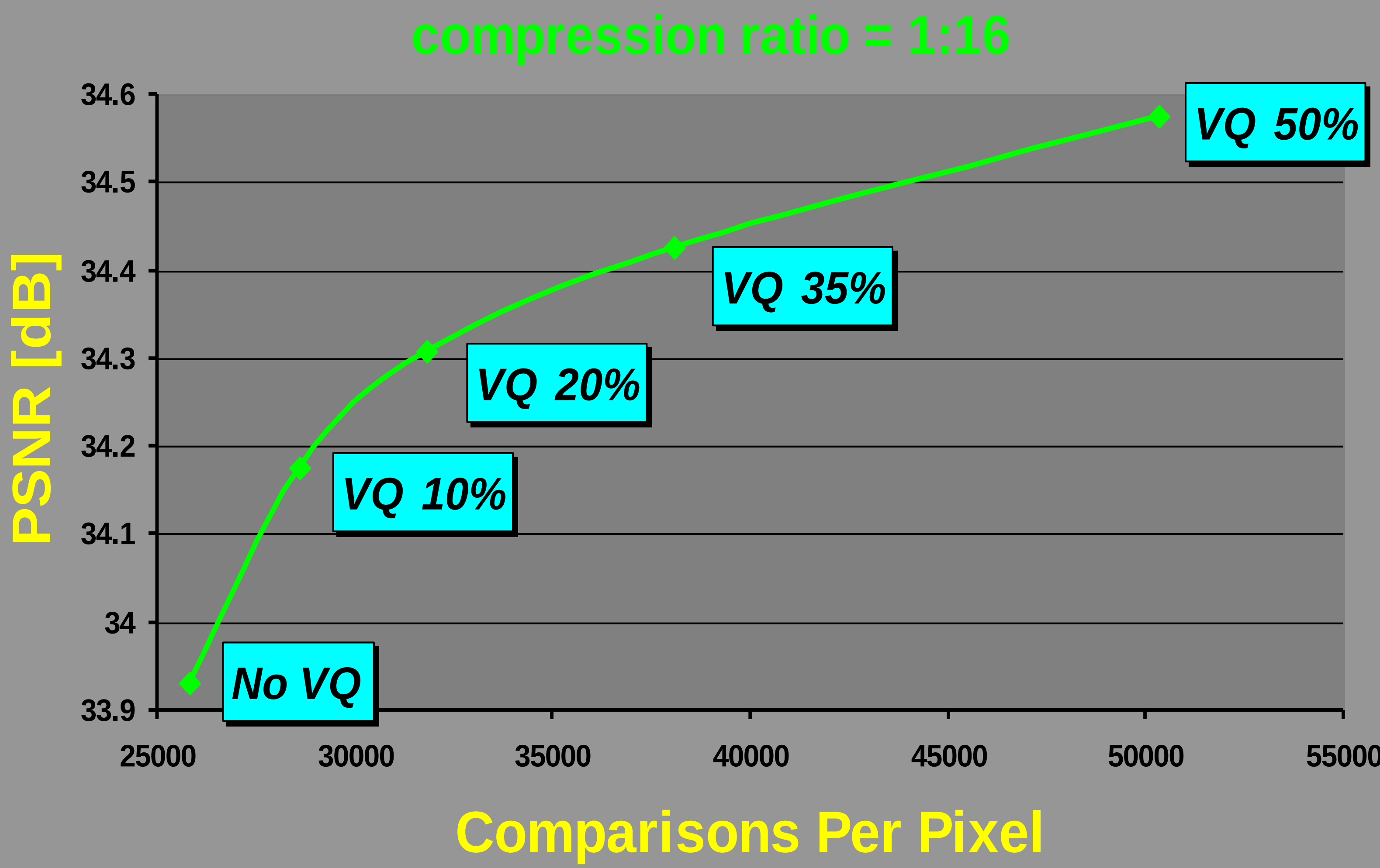
Hybrid Fractal - VQ Coding (cont'd)

Examined with the help of:

Yaniv Gur & Alex Trigov



Hybrid Fractal - VQ Coding (cont'd)



Entropy Coding

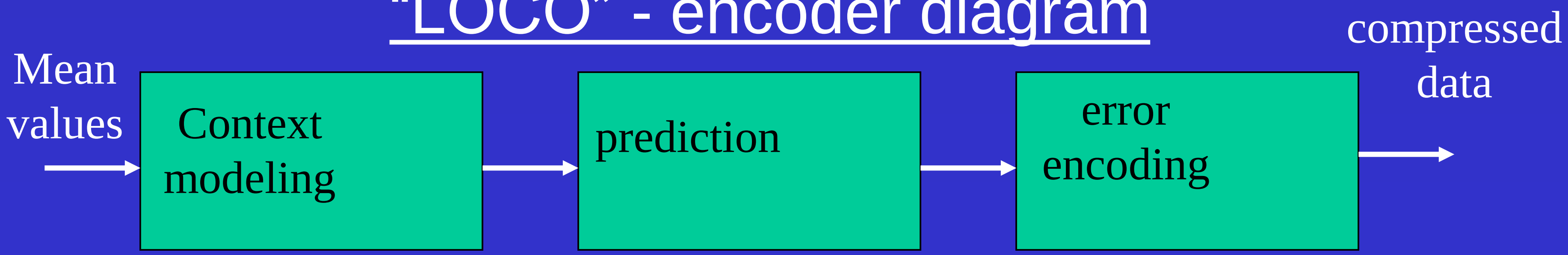
Entropy coding of "Fractal code" : $W = \cup \{ \overline{R}_i, s_i, j_i \}$

Typical Bit allocation: Range mean - 7 bits
Scale factor - 5 bits
Domain index ≈ 12 bits

Entropy coding of scale and index values achieves only 5% reduction in the number of bits used (≈0.85 bit).

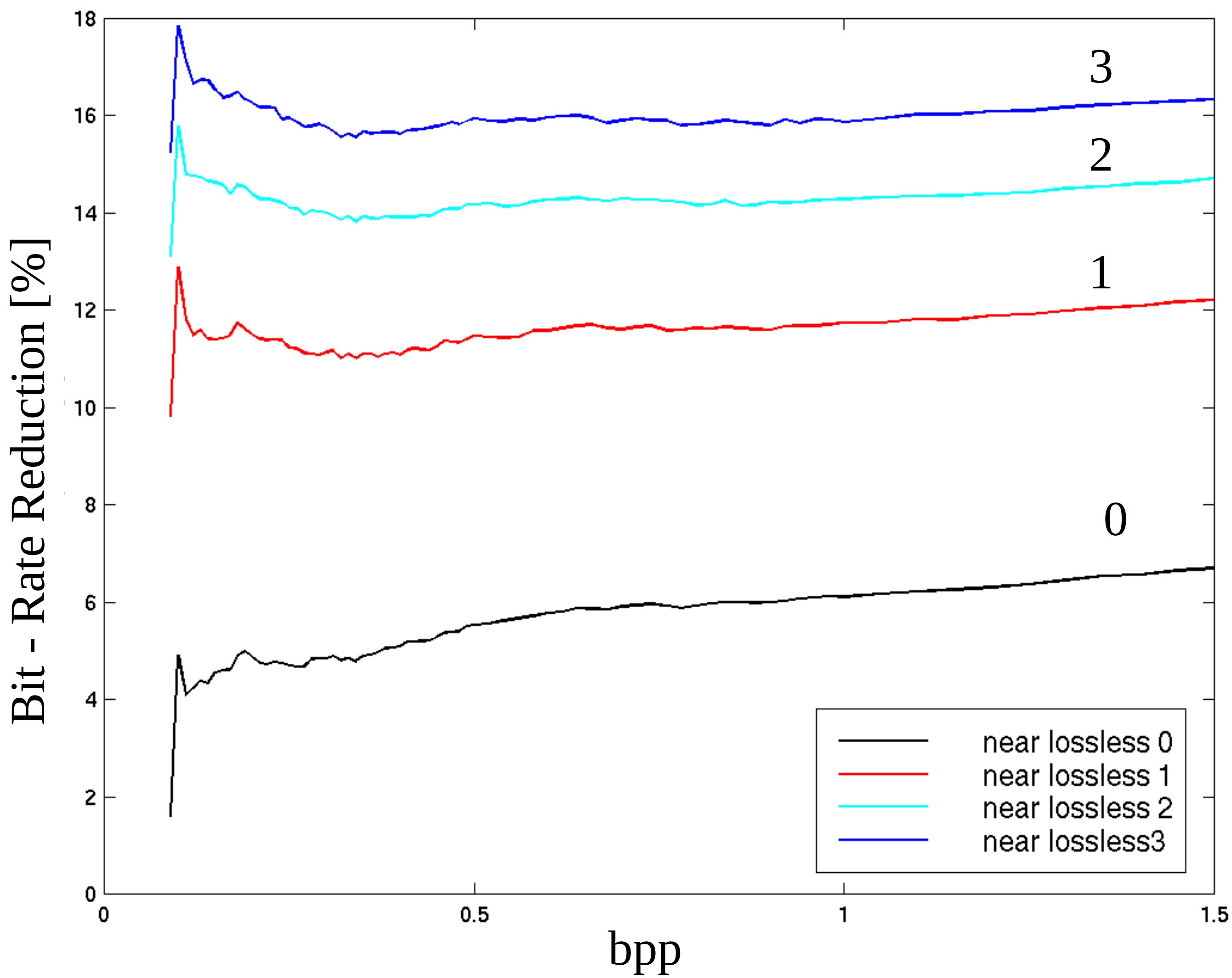
The mean values of neighboring range blocks are highly correlated (3 bits reduction is achievable using "LOCO")

"LOCO" - encoder diagram



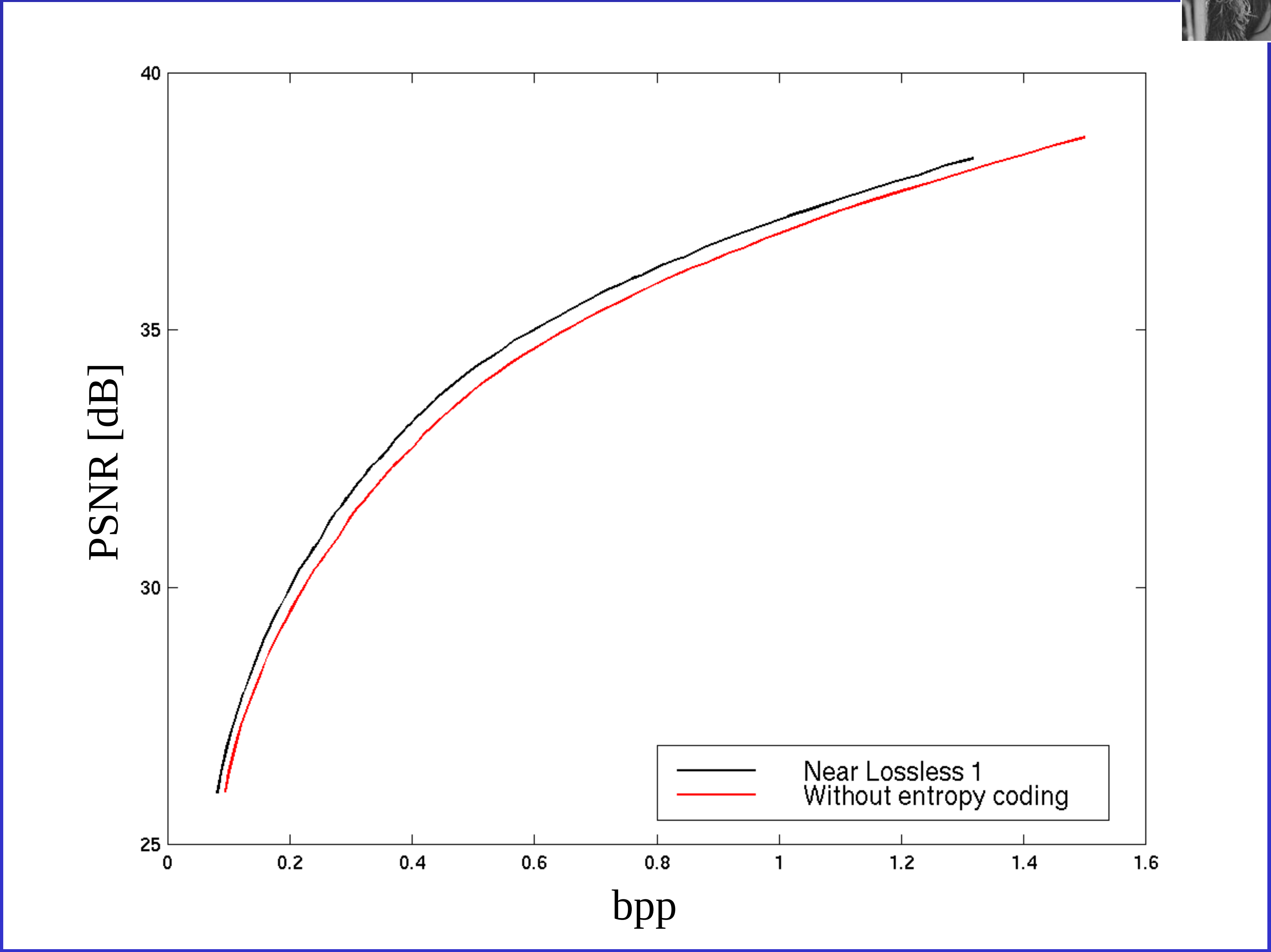
Entropy Coding - Results

Size of range blocks : 16,8,4 & 2



Entropy Coding - Results (cont'd)

Size of range blocks : 16,8,4 & 2



Summary and Conclusions

- Rate-Distortion - reduction of reconstruction error
- Collage error-Computational complexity - reduction of computational complexity
- Adaptive coding combining Rate-Distortion and Collage-Complexity - enables direct control of rate and complexity
- Splitting without search
- “Fast search” methods

Summary and Conclusions (cont'd)

- Matching pursuit combined with Rate-Distortion splitting criterion - reduction of reconstruction error and complexity
- VQ combined with Rate-Distortion splitting criterion - Better results than matching pursuit at the same bit rate and complexity.
- Entropy coding - =12% bit rate reduction

THE END