Bounds on the Channel Distortion of Vector Quantizers

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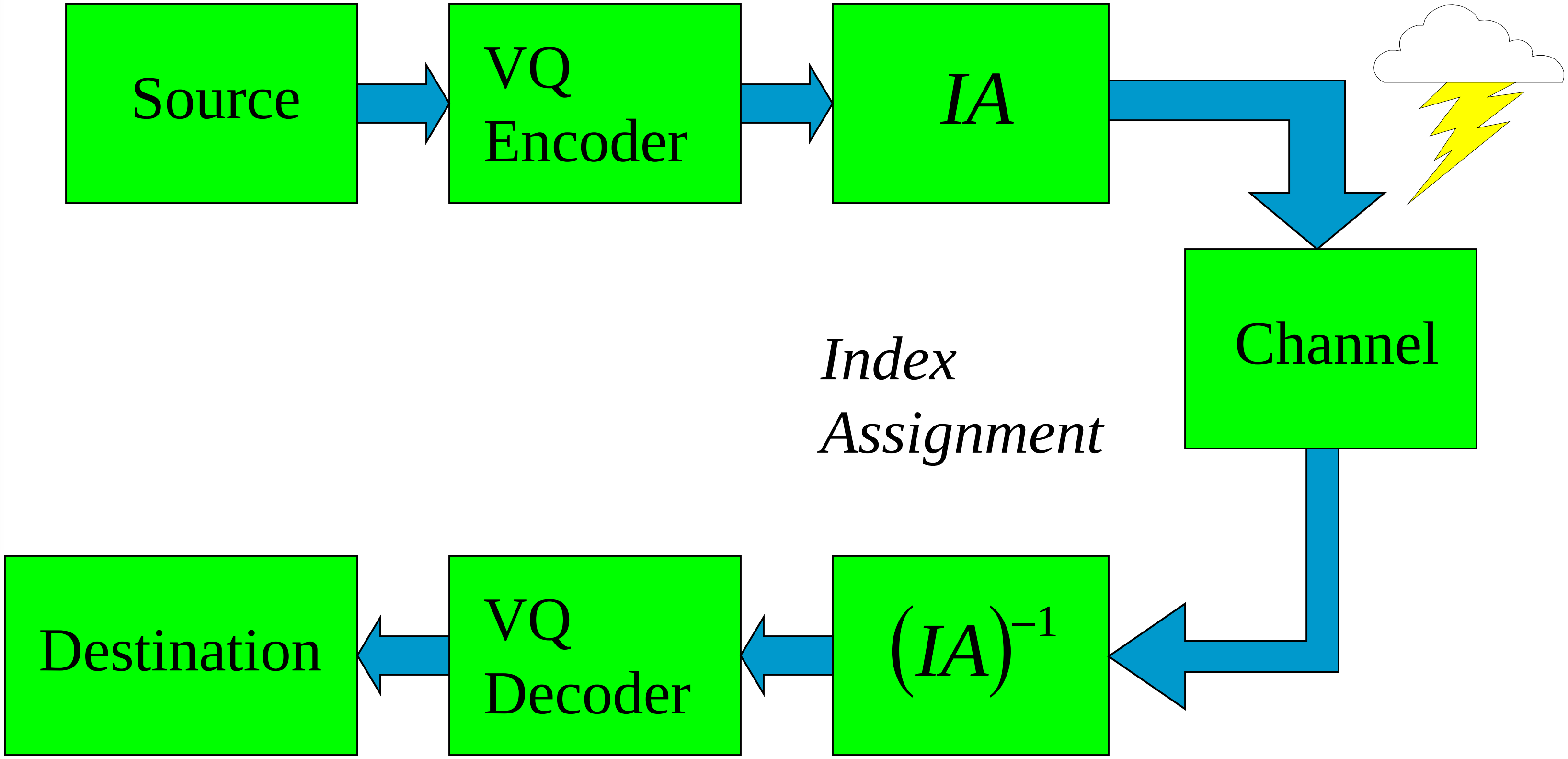
Outline

- Vector quantization
- Definition of channel distortion
- Index assignment
- Performance bounds
- Average performance over all index assignments
- Numerical results
- Conclusions

Vector Quantizer (VQ) Based Communication System



Index Assignment

Vector Quantization

- Widely used method for low-bit-rate communication
- The *signal space* (Ω) of all possible source-vectors is divided into non-overlapping *regions* (R_i)
- Each region is represented by a *codevector* (ϕ_i)
- Codebook – The collection of all codevectors
- Codevector indices are sent through the channel

Notation

Ω - Entire Signal Space

$p(x)$ - Probability density function of source vector x

R_i - Partition region i (of N)

$p_i = \int_{R_i} p(\underline{x}) \cdot d\underline{x}$ - Probability of region i

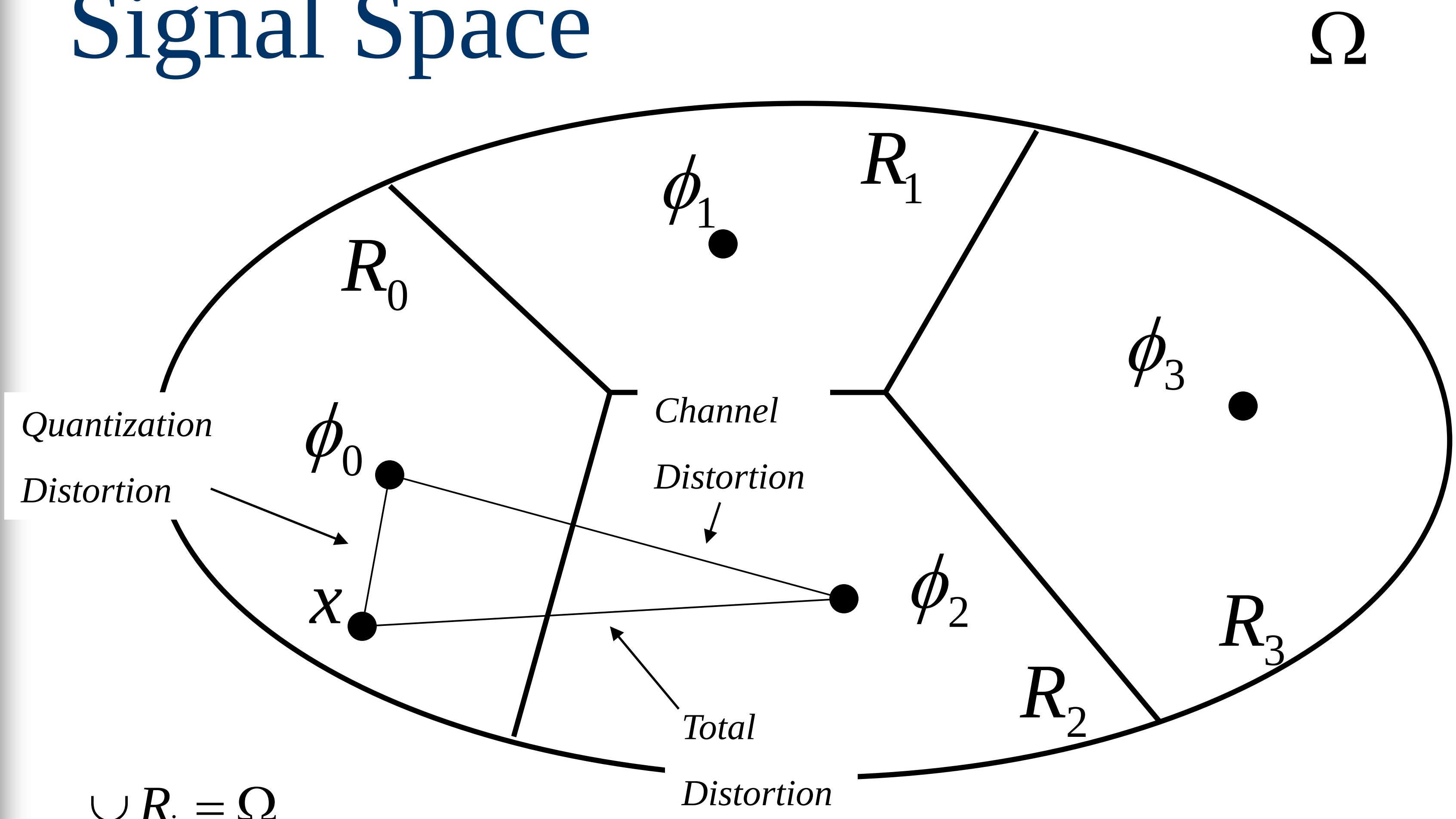
ϕ_i - Codevector i

$d(\cdot,\cdot)$ - Distance Measure

π - Permutation Matrix

Q - Channel Transition Matrix

Signal Space



$$\bigcup_i R_i = \Omega$$

$$R_i \cap R_j = \varnothing$$

Distortion Values - I

Total (overall) distortion

$$D_T = E\left[d\left(\underline{x},\hat{\underline{x}}\right)\right] = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{\pi\cdot Q\cdot\pi^T\right\}_{ij}\int_{R_i} d\left(\underline{x},\underline{\phi}_j\right)\cdot p\left(\underline{x}\right)\cdot d\,\underline{x}$$

Quantization distortion

$$D_{Q=I} = E\left[d\left(\underline{x},\hat{\underline{x}}\right)\right]\Big|_{Q=I} = \sum_{i=0}^{N-1}\int_{R_i} d\left(\underline{x},\underline{\phi}_i\right)\cdot p\left(\underline{x}\right)\cdot d\,\underline{x}$$

The partition regions and codevectors are designed to minimize the quantization distortion

Distortion Values - II

Channel distortion

$$D_C = \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1} \left\{\pi \cdot Q \cdot \pi^T\right\}_{ij} \cdot d\left(\underline{\phi}_i, \underline{\phi}_j\right) = trace\left\{P \pi Q \pi^T D\right\}$$

where

$P = diag\left\{p_0, p_1, \ldots, p_{N-1}\right\}$ - Partition regions probability matrix

$\left\{D\right\}_{ij} = d\left(\underline{\phi}_i, \underline{\phi}_j\right)$ - codevectors distance matrix

For the Euclidean distance measure and Centroid Quantizers

Total distortion = Quantization distortion + Channel distortion

$$D_T = D_{Q=I} + D_C$$

Index Assignment

- Assignment of indices to codevectros affects system performance under channel errors
- There are *N*! possible index assignments
- Looking for the best assignment is a *Quadratic Assignment* problem and is known to be NP-complete
- Various <u>sub-optimal</u> index assignment algorithms are known – Local index switching, Genetic algorithms, Simulated annealing

Motivation for Determining Performance Bounds

- Difficulty in obtaining good assignments
- Need to estimate the performance of given assignments as compared to all possible index assignments
- Evaluate possible “Assignment Gain” when searching for good assignments

Bounds Outline - I

Channel Distortion

$$D_C = \frac{1}{2} trace\left\{Q\pi^T \hat{D}\pi\right\} \text{ where } \hat{D} = DP + P^T D^T$$

Define

$$s_i = \sum_{j=0}^{N-1} \hat{D}_{ij} \quad \text{and} \quad k = \arg\max\left\{s_i\right\}$$

$$C_i = \begin{bmatrix} 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \end{bmatrix}$$

$$\uparrow$$

$$i - \text{th column}$$

Bounds Outline - II

Define

$$\alpha_i = (s_k - s_i)/N$$

$$S = \sum_{i=0}^{N-1} \alpha_i$$

$$\tilde{D} = \hat{D} + \sum_{i=0}^{N-1} \alpha_i \left(C_i + C_i^T\right)$$

λ_i - Eigenvalues of the channel transition matrix Q (descending order)

ω_i - Eigenvalues of the matrix $\tilde{D}$ (descending order)

Lower and Upper Bounds

$$\frac{1}{2}\left(\lambda_0\omega_0 + \sum_{i=1}^{N-1} \lambda_i \cdot \omega_{N-i}\right) - S \le D_C \le \frac{1}{2}\left(\lambda_0\omega_0 + \sum_{i=1}^{N-1} \lambda_i \cdot \omega_i\right) - S$$

Complexity – Finding the eingenvalues of two matrices

Average Performance

A related expression for the average performance over all possible index assignment

$$\langle D_C \rangle = \frac{1}{2}\lambda_0\omega_0 + \frac{1}{2(N-1)}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j - S$$

May also help in finding how well a given assignment performs

Special Cases and Numerical Results

- The proposed bounds were compared to the average performance as well as to “good“ and “bad” assignments found in simulations
- For 3-bit quantizers, all assignment were checked by exhaustive search
- For 4-bit and larger quantizers, “Good” (“bad”) assignments were found by a index switching algorithm (local optimization)

Uniform Scalar Quantizer and a Uniform Source Under the BSC

$$\frac{2(N-1)(N+1)}{3N^2}2q \le D_C \le \frac{2(N-1)(N+1)}{3N^2}\left[1-(1-2q)^L\right]$$

$$\langle D_C \rangle = \frac{2N(N+1)}{3N^2}\left[1-(1-q)^L\right]$$

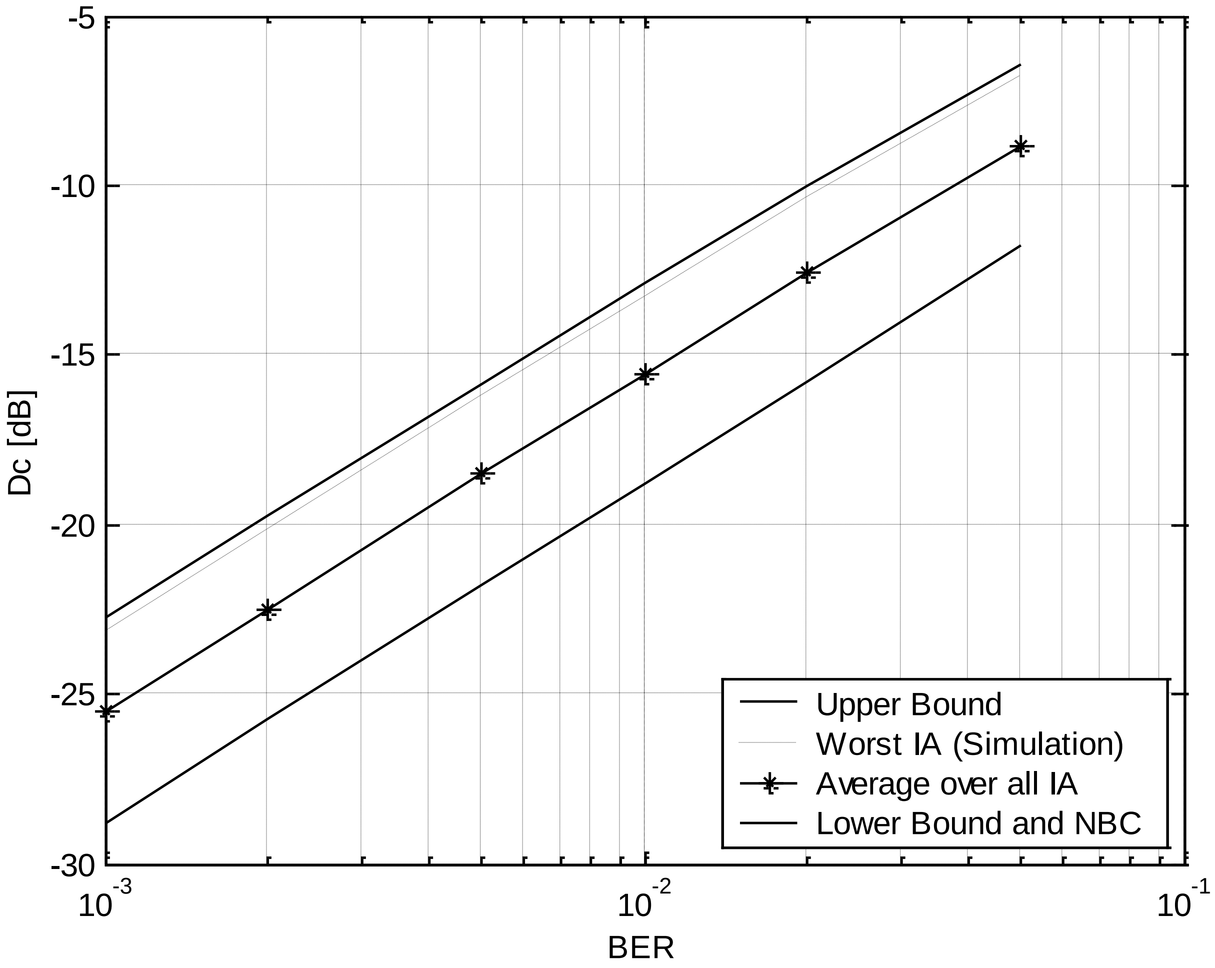
where

N - # of quantization levels

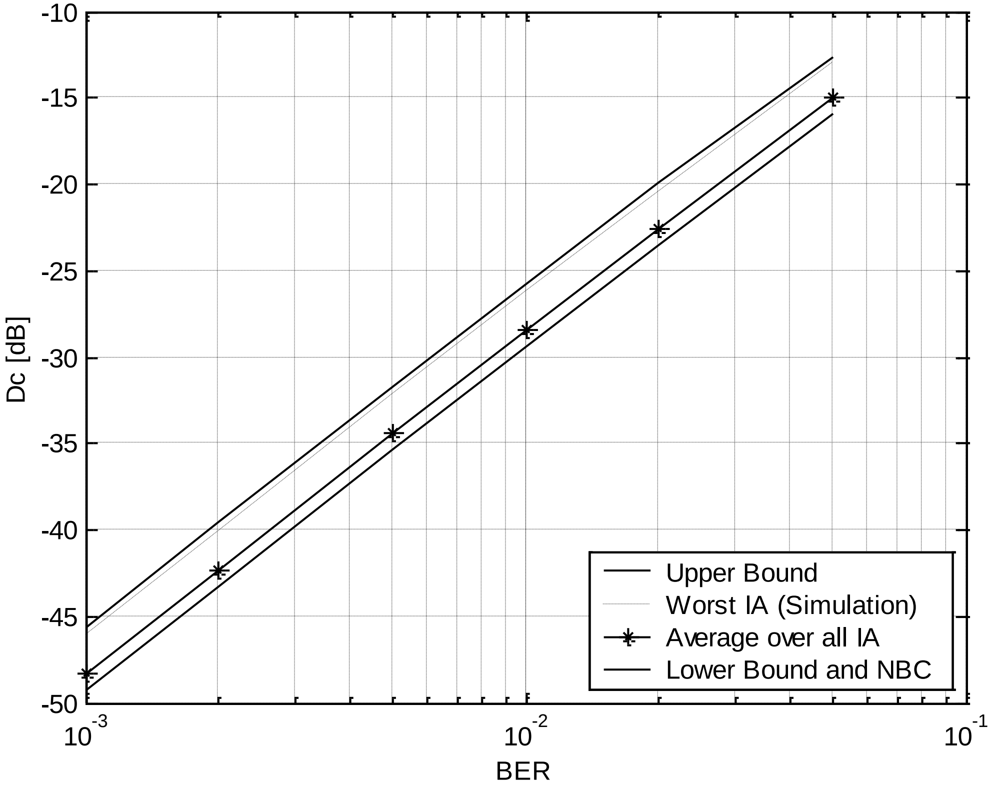
q - Bit Error Rate (BER)

The lower bound coincide with the performance of the Natural Binary Code

4-bit Uniform Scalar Quantizer and a Uniform Source Under the BSC

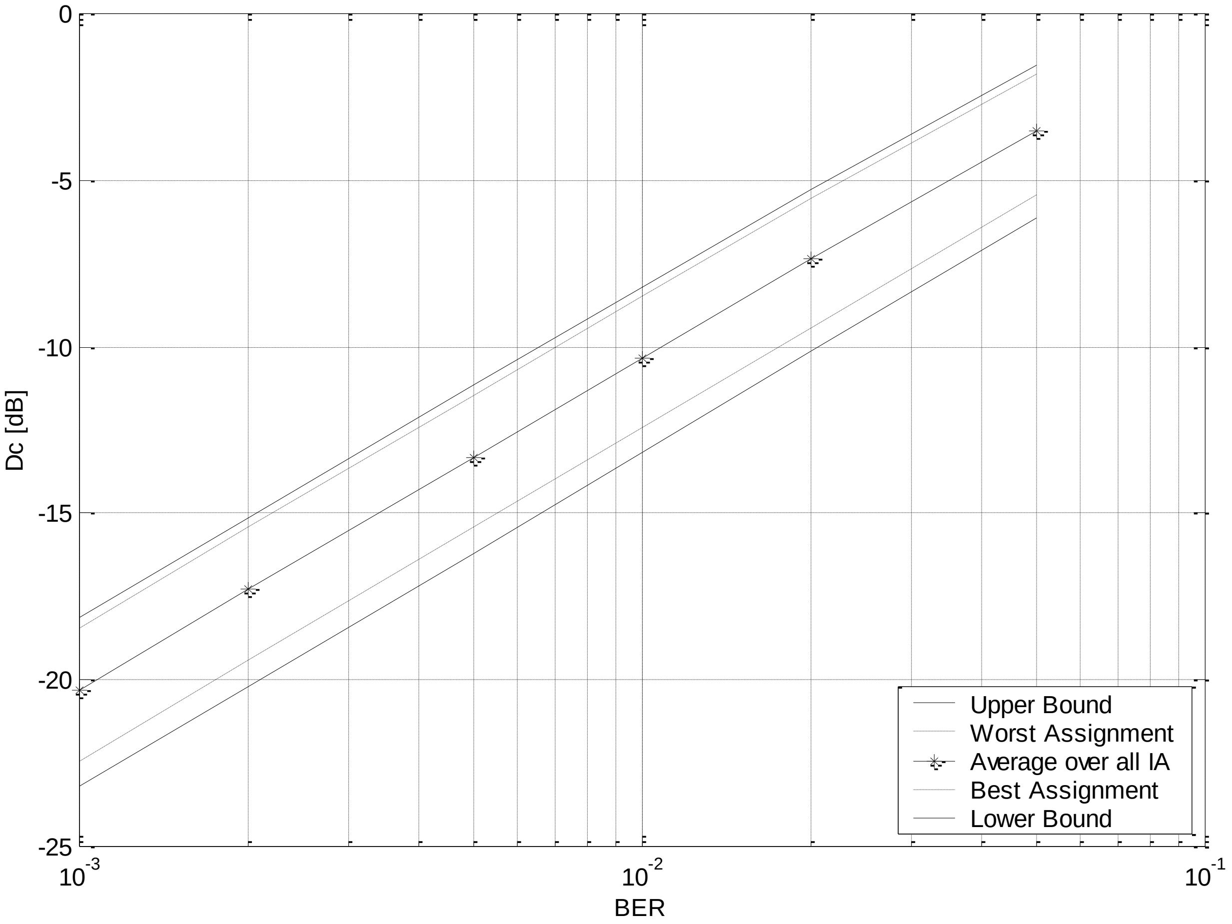


A 4-bit Uniform Quantizer and a Uniform Source Using a (7,4) Hamming Error-Correcting-Code Under the BSC

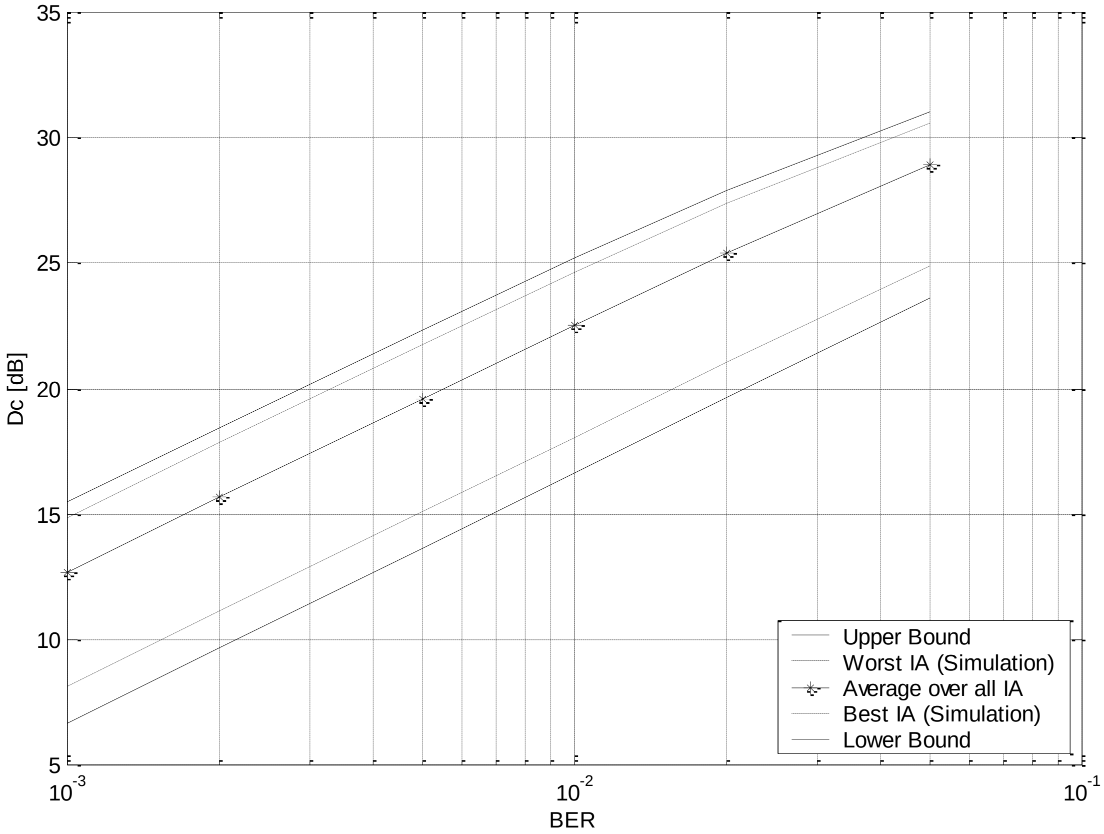


The implementation of the channel protection brought the bounds closer together, decreasing the importance of index assignment.

3-bit PDF-Optimized Scalar Quantizer for Gaussian Source Under the BSC



Three-Dimensional, 8-bit PDF-Optimized Vector Quantizer for Palette Limited Images Using the L*a*b* Color Space



Conclusions

- Upper and lower bounds of the distortion due to channel errors for vector quantizers, over all possible index assignments were introduced.
- Related expression for the average performance was shown
- Results enable the VQ designer to
  - Estimate the gain that may be obtained by a search for an efficient index assignment
  - Estimate the performance of a given index assignment as compared to all possible assignments.
- Bounds are reasonably close to the performance of the assignments found in simulations.