SPEECH ENHANCEMENT USING OPTIMAL NON-LINEAR SPECTRAL AMPLITUDE ESTIMATION

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ABSTRACT

A speech enhancement system which utilizes an optimal (in the minimum mean square error sense) short-time spectral amplitude estimator is described. The derivation of the optimal estimator is based on modeling speech as a quasi-periodic signal, and on applying spectral decomposition. The optimal spectral amplitude estimator and a recently developed vector spectral subtraction amplitude estimator, are found to be nearly equivalent. The optimal spectral amplitude estimator coincides with a Wiener spectral amplitude estimator at high signal to noise ratio (SNR) values, and is found to be superior to it at low SNR values.

The enhanced speech obtained by using the proposed system, is less spectrally distorted, although contains some more residual noise, than the enhanced speech obtained by using the Wiener spectral amplitude estimator, in the same system. In addition, it is free of the "musical noise" characteristic to the spectral subtraction algorithm. Both systems, the proposed one and spectral subtraction, have approximately the same complexity.

I. INTRODUCTION

In this paper we describe an algorithm for enhancing speech degraded by statistically independent additive noise, using only the noise corrupted speech signal. This algorithm capitalizes on the major importance of the short-time spectral amplitude, relative to the short-time phase, in speech perception, and focuses on its optimal estimation. For reconstructing the enhanced speech signal, the estimated spectral amplitude is combined with the phase of the degraded speech.

We base the estimation on modeling speech as a quasi-periodic signal, and apply spectral decomposition. Thus, to a good approximation, the estimation problem can be formulated as that of estimating the amplitude of a sinusoid corrupted by additive noise. The amplitude estimator derived here is optimal in the minimum mean square error (m.m.s.e) sense. Interestingly, it was found that the optimal spectral amplitude estimator, and a recently developed vector spectral subtraction amplitude estimator [1], are nearly equivalent. In addition, the optimal spectral amplitude estimator coincides with a Wiener spectral amplitude estimator, at high signal to noise ratio (SNR) values, and is found to be superior to it at low SNR values.

The enhanced speech obtained by using the proposed system, suffers less spectral distortion, although contains some more residual noise, than the enhanced speech obtained by using the Wiener spectral amplitude estimator, in the same system. In addition, it is free of the "musical noise" characteristic to the spectral subtraction algorithm.

The paper is organized as follows: In Section II we derive the optimal spectral amplitude estimator, and discuss its properties. In Section III we describe the implementation of the optimal spectral amplitude estimator in a speech enhancement system, and discuss its performance. In Section IV we summarize the paper and draw conclusions.

II. OPTIMAL SHORT-TIME SPECTRAL AMPLITUDE ESTIMATOR

In this section we derive the optimal m.m.s.e short-time spectral amplitude estimator. On the basis of modeling speech as a quasi-periodic signal, and by applying spectral decomposition, the estimation problem is formulated as that of estimating the amplitude of a sinusoid corrupted by additive noise. Let $y(n)$ denote the observed signal:

$$y(n) = A\cos(\omega^* n + \varphi) + d(n) \qquad (1)$$

and assume the following assumptions: A is a Rayleigh distributed random variable (r.v.) with parameter σ_A; ω^* is a uniformly distributed r.v. on $[\omega_k-\Omega,\ \omega_k+\Omega]$, where ω_k denotes the center of a frequency band of width 2Ω; φ is a uniformly distributed r.v. on $[0,2\pi]$; and d_n is a zero mean stationary gaussian noise with a given power spectral density $S_d(\omega)$. We assume also that A, ω^*, φ, and $d(n)$ are statistically independent.

Spectral decomposition can be efficiently done by means of the short-time Fourier transform (STFT) [2]. This is equivalent to passing the signal through a bank of N quadrature demodulators, with identical low pass filters, and modulation frequencies (in radians) of $\omega_l = 2\pi l / N$, $l=0,...,N-1$. Assuming an ideal low pass filter $h(n)$, with cutoff frequency at Ω radians, and considering the relevant output (say, from the k-th quadrature demodulator), we get the following complex representation of (1):

$$Y_n = A\exp[j(\omega_\Delta n + \varphi)] + D_n \qquad \textbf{(2-a)}$$

$$\stackrel{\Delta}{=} R_n \exp(j\vartheta_n) \qquad (2\text{-b})$$

where, $\omega_\Delta \stackrel{\Delta}{=} \omega^* - \omega_k$ is a uniformly distributed r.v. on $[-\Omega,\Omega]$, and D_n is the complex envelope of the noise in the frequency band centered at ω_k. D_n is a zero mean complex gaussian process, whose variance $2\sigma_d^2$ equals to:

$$2\sigma_d^2 \stackrel{\Delta}{=} E\{|D_n|^2\} \qquad (3)$$

$$= \int_{-\pi}^{\pi} S_d(\omega+\omega_k)\,|H(\omega)|^2\ \frac{d\omega}{2\pi}$$

where $H(\omega)$ is the Fourier transform of the low pass filter unit sample response $h(n)$.

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The optimal m.m.s.e. amplitude estimator $\hat{A}_{opt}$ of A, given the observation (R_n, ϑ_n), is:

$$\hat{A}_{opt} = E\{A \mid R_n, \vartheta_n\} \tag{4}$$

$$= \frac{\int_0^{\infty}\int_{-\Omega}^{\Omega}\int_0^{2\pi} a f(r_n, \vartheta_n \mid a, \omega_\Delta, \varphi)\, f(a, \omega_\Delta, \varphi)\, d\varphi\, d\omega_\Delta\, da}{\int_0^{\infty}\int_{-\Omega}^{\Omega}\int_0^{2\pi} f(r_n, \vartheta_n \mid a, \omega_\Delta, \varphi)\, f(a, \omega_\Delta, \varphi)\, d\varphi\, d\omega_\Delta\, da}$$

where, $E\{\cdot\}$ denotes the expectation operator; r_n, ϑ_n, a, ω_Δ, and φ denote the realizations of the random variables R_n, ϑ_n, A, ω_Δ, and φ, respectively; $f(r_n, \vartheta_n \mid a, \omega_\Delta, \varphi)$ is the conditional probability density function (PDF) of (R_n, ϑ_n), given A, ω_Δ, and φ; $f(a, \omega_\Delta, \varphi)$ is the common PDF of A, ω_Δ, and φ. Since the real and imaginary parts of D_n are zero mean statistically independent gaussian random variables, and have the same variance σ_d^2, $f(r_n, \vartheta_n \mid a, \omega_\Delta, \varphi)$ is given by:

$$f(r_n, \vartheta_n \mid a, \omega_\Delta, \varphi) = \frac{r_n}{2\pi\sigma_d^2}\exp\{\frac{-1}{2\sigma_d^2}\left|r_n e^{j\vartheta_n} - a e^{j(\omega_\Delta n + \varphi)}\right|^2\} \tag{5}$$

Because A, ω_Δ, and φ are assumed to be statistically independent, $f(a, \omega_\Delta, \varphi)$ can be factored into $f(a) f(\omega_\Delta) f(\varphi)$, and hence is given by:

$$f(a, \omega_\Delta, \varphi) = \begin{cases} \dfrac{a}{\sigma_A^2}\exp(-\dfrac{a^2}{2\sigma_A^2})\dfrac{1}{2\Omega}\,\dfrac{1}{2\pi} & \begin{matrix} 0 \le a < \infty \\ -\Omega \le \omega_\Delta \le \Omega \\ 0 \le \varphi \le 2\pi \end{matrix} \\ 0 & otherwise \end{cases} \tag{6}$$

Defining an a-priori SNR, γ_A, by $E[A^2]/2\sigma_d^2$, and an a-posteriori SNR, γ_n, by $R_n^2/2\sigma_d^2$, and substituting (5) and (6) into (4), we get (see appendix A):

$$\hat{A}_{opt} = \Gamma(1.5)\sqrt{\frac{v_n}{\gamma_n^2}}\exp(-\frac{v_n}{2})[(1+v_n)I_0(\frac{v_n}{2}) + v_n I_1(\frac{v_n}{2})]R_n \tag{7}$$

where, v_n is defined by:

$$v_n \overset{\Delta}{=} \frac{\gamma_A}{1+\gamma_A}\gamma_n \tag{8}$$

$\Gamma(\cdot)$ is the Gamma function ($\Gamma(1.5) = \sqrt{\pi}/2$), $I_0(\cdot)$ and $I_1(\cdot)$ are the modified Bessel functions of zero and first order, respectively. Note also that since A is a Rayleigh distributed r.v., $\gamma_A = \sigma_A^2/\sigma_d^2$.

$\hat{A}_{opt}$ given by (7), is seen to be obtained from R_n by a multiplicative non-linear gain function, which is defined by: $G_{opt}(\gamma_A, \gamma_n) \overset{\Delta}{=} \hat{A}_{opt}/R_n$. $G_{opt}(\gamma_A, \gamma_n)$ depends on the a-priori and a-posteriori SNR values, γ_A and γ_n, respectively, and is conveniently described by a set of parametric gain curves [3,4]. The gain curves which result from (7) are shown in Fig. 1. $\gamma_n - 1$ in Fig. 1 is interpreted as the "measured SNR", since $\gamma_n = R_n^2/2\sigma_d^2$, and R_n equals to the length of the signal plus noise resultant vector (see (2)).

The curves in Fig. 1 show an increase in gain as the a-posteriori SNR γ_n decreases, while keeping the a-priori SNR γ_A constant. This is a direct consequence of incorporating the a-priori SNR in the amplitude estimation process. For a given γ_A, which results from specific values of σ_A^2 and σ_d^2, γ_n is proportional to R_n. Therefore, decreasing γ_n means decreasing R_n, and an increase of the gain is expected for a correct estimation of A.

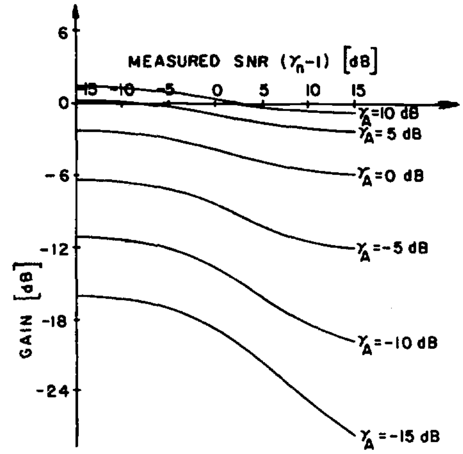

Fig. 1. Parametric gain curves, describing $G_{opt}(\gamma_A, \gamma_n)$.

It is of interest to note that the gain curves obtained in the optimal spectral amplitude estimation, and the gain curves obtained in the vector spectral subtraction amplitude estimation [1], coincide in the shown range. This fact implies that the vector spectral subtraction amplitude estimator is optimal in the m.m.s.e. sense.

The asymptotic behavior of $\hat{A}_{opt}$ at high SNR values (i.e., $v_n \to \infty$), is easily obtained by considering the relationship $I_1(v_n) = \partial I_0(v_n)/\partial v_n$, and the following approximation for $I_0(v_n)$ [3]:

$$I_0(v_n) \simeq \frac{\exp(v_n)}{\sqrt{2\pi v_n}} \qquad v_n >> 1 \tag{9}$$

We get then from (7)

$$\hat{A}_{opt} \simeq \frac{\gamma_A}{1+\gamma_A}R_n \tag{10}$$

Since we finally estimate the spectral component $A\exp[j(\omega_\Delta n + \varphi)]$ by $\hat{A}_{opt}\exp(j\vartheta_n)$, where ϑ_n is the noisy phase (see (2-b)), (10) means that at high SNR values, the estimator of $A\exp[j(\omega_\Delta n + \varphi)]$ approaches the optimal linear (Wiener) estimator. Therefore, in the sequel, (10) is interpreted as a Wiener spectral amplitude estimator. The gain function $G_W(\gamma_A, \gamma_n)$, which results from (10), is independent of γ_n. Each of its corresponding gain curves, is a horizontal line at a level of $20\log(\gamma_A/1+\gamma_A)$.

The spectral amplitude estimators given by (7) and (10), were tested and compared in estimating the amplitude of a complex sinusoid, buried in a complex zero mean white noise. The a-priori SNR γ_A ranged from -5 dB to 10 dB, and was assumed to be known. The noise variance $2\sigma_d^2$, corresponding to each γ_A value, is assumed to be known as well. An ensemble of 20480 observations, matched to the signal model in (2), and containing twenty different realizations of the pair (A, φ), was used for each a-priori SNR value. The normalized (by the variance of A) residual mean square error (MSE), obtained in this experiment, is described in Fig. 2. This figure demonstrates the superiority of the optimal spectral amplitude estimator, especially at low SNR values, if the a-priori SNR value and the noise variance are known. Of course it is useless, in this experiment, to use the optimal estimator for $\gamma_A \le 1dB$, or the Wiener estimator for $\gamma_A \le 2dB$, since the resulting MSE exceeds the variance of A. However, in practice we do not know the expected value of A exactly, and therefore we use the derived estimators for any value of γ_A.

The performance shown in Fig. 2 represents the

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best performance one can get from the examined estimators, since γ_n and γ_A were known exactly. In practice, γ_n and γ_A are unknown, and estimates of their values are used. Therefore, the performance of the examined estimators depends on how well γ_n and especially γ_A are estimated. This problem is considered further in Section III.

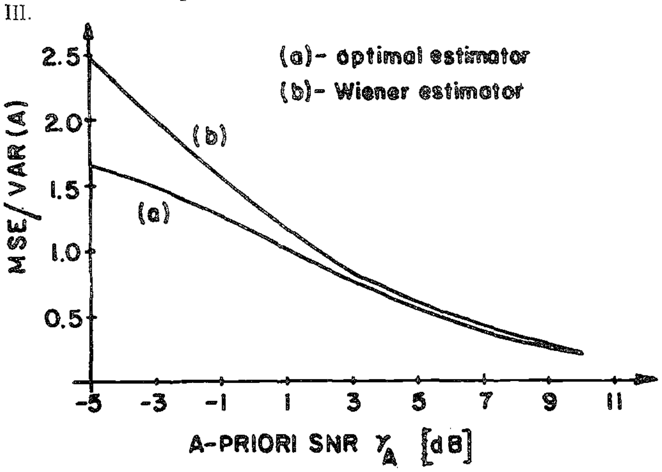

Fig. 2. Performance comparison of the optimal and the Wiener spectral amplitude estimators.

We noticed in our experiments that the performance of the vector spectral subtraction amplitude estimator [1], as measured by a similar experiment, is nearly equivalent to the performance obtained with the optimal spectral amplitude estimator, which is shown by line (a) in Fig. 2. This fact reconfirms our previous conclusion, concerning the optimality of the vector spectral subtraction amplitude estimator.

III. SPEECH ENHANCEMENT SYSTEM DESCRIPTION

The optimal spectral amplitude estimator derived in Section II, was embedded in a speech enhancement system which is described in this section. The noisy speech to be enhanced is first bandlimited to 0.2-3.2 kHz, and then sampled at 8 kHz. Each analysis frame, which contains 256 samples of the noisy speech and overlaps the previous analysis frame by 192 samples, is spectrally decomposed by means of STFT analysis [2], using a Hanning window. Each STFT sample is modified by the multiplicative gain function $G_{opt}(\gamma_A,\gamma_n)$, after estimating its a-priori and a-posteriori SNR values, γ_A and γ_n, respectively. The modified STFT samples are used for synthesizing the enhanced speech, by using the well known overlap and add method [2]. Since $G_{opt}(\gamma_A,\gamma_n)$ is a real valued function, the multiplicative modification of each STFT sample made by $G_{opt}(\gamma_A,\gamma_n)$, is equivalent to estimating its absolute value, and using its noisy phase. In the proposed system, a look-up table which contains discrete values of the gain function $G_{opt}(\gamma_A,\gamma_n)$ is used. $G_{opt}(\gamma_A,\gamma_n)$ was calculated for 961 pairs of (γ_A,γ_n-1) values, which equally divide the square region [-15:15, -15:15]dB. It was judged by informal listening, that using discrete values of the gain function in the above range, rather than recalculating it for each estimated value of the pair (γ_A,γ_n-1), appears harmless to the enhanced speech quality.

A crucial issue for a successful implementation of the optimal spectral amplitude estimator, is how well can the a-priori and the a-posteriori SNR values, γ_A and γ_n, respectively, be estimated from the noisy data. To estimate γ_A and γ_n, recall that $\gamma_A \overset{\Delta}{=} E[A^2]/2\sigma_d^2$, and $\gamma_n \overset{\Delta}{=} R_n^2/2\sigma_d^2$. Since the problem of estimating the noise variance from non-speech intervals is well treated in the literature (e.g., in [3]), we will not deal with it here and assume σ_d^2 to be known. The problem of estimating $E\{A^2\}$, is the same problem which arises in Wiener filtering, where the spectrum of the desired signal is assumed to be a-priori known. Lim & Oppenheim [4] suggested several solutions to this problem. However, none of these solutions provided adequate performance when implemented in the proposed system. We found that a "decision-directed" approach for estimating the expected value of A^2 is useful in the proposed system. Specifically, let $\gamma_A(n)$ and $\hat{A}_n$ denote the estimated values of γ_A and A, respectively, in the n-th frame and a given frequency band. The proposed estimator $\hat{\gamma}_A(n)$ is a weighted sum of $\hat{A}_{n-1}^2/2\sigma_d^2$ and the "measured SNR". That is,

$$\hat{\gamma}_A(n) = \alpha\hat{A}_{n-1}^2/2\sigma_d^2+(1-\alpha)(\gamma_n-1) \qquad 0\leq\alpha\leq1 \tag{11}$$

$\alpha\hat{A}_{n-1}^2/2\sigma_d^2$ can be considered as the predicted value of $\gamma_A(n)$, based on previous estimated values of A. Since $\gamma_A=E\{\gamma_n\}-1$ (follows directly from (2) and the definitions of γ_A and γ_n), $(1-\alpha)(\gamma_n-1)$ reflects the contribution of the present observation to the estimate of $\gamma_A(n)$. Based on informal listening, we recommend using α=0.97 in (11). Although $1-\alpha$ equals only 0.03, the term $(1-\alpha)(\gamma_n-1)$ in (11) was found to be important, and contributes to the crispness of the enhanced speech.

It is worthwhile to note that using α=0 in (11), (i.e., $\hat{\gamma}_A(n)=\gamma_n-1$) and the gain function $G_{opt}(\gamma_A,\gamma_n)$, results in a single gain curve, which is very close to the gain curve shown in [3] for the spectral subtraction algorithm. These two gain curves are shown in Fig. 3.

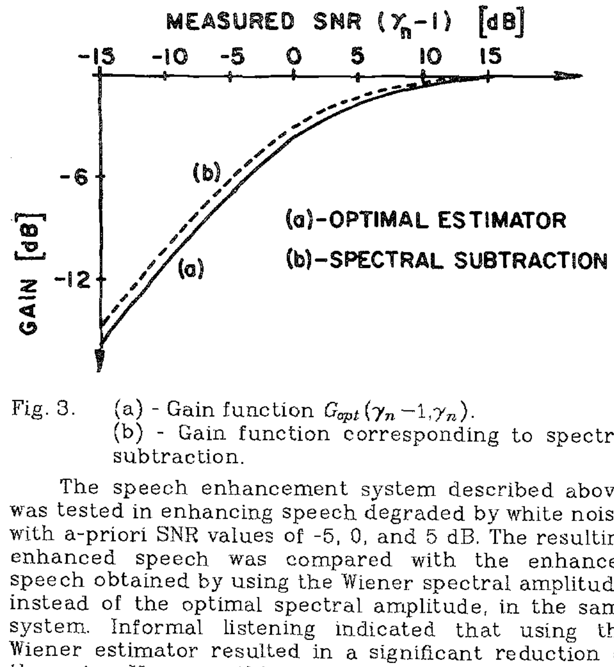

Fig. 3. (a) - Gain function $G_{opt}(\gamma_n-1,\gamma_n)$.
(b) - Gain function corresponding to spectral subtraction.

The speech enhancement system described above, was tested in enhancing speech degraded by white noise, with a-priori SNR values of -5, 0, and 5 dB. The resulting enhanced speech was compared with the enhanced speech obtained by using the Wiener spectral amplitude, instead of the optimal spectral amplitude, in the same system. Informal listening indicated that using the Wiener estimator resulted in a significant reduction of the noise. However, this was accompanied with a noticeable spectral distortion in the enhanced signal, which became quite severe at low SNR values (0 dB and below). With the optimal spectral amplitude estimator, some residual colorless noise remains, but the enhanced speech is less distorted, especially at low SNR values.

Table I presents a performance comparison of the examined algorithms, by means of a spectral log-segmental SNR measure. The spectral log-segmental SNR values measured for the input noisy speech are also included. This measure is the mean (in dB) SNR value, based on calculating the SNR in critical bands of each analyzed frame. The results shown in Table I clearly demonstrate the superiority of the optimal estimator. However, they seem to contradict our previous conclusion, concerning the coinciding of the Wiener and the optimal spectral amplitude estimators at high SNR values. But, we recall that this happened when γ_A was known and not estimated as it is done here.

total SNR[dB]	noisy speech	Wiener	optimal
5	9.39	12.42	15.36
0	5.27	9.31	12.02
-5	2.75	6.83	8.89

Table I: Performance comparison by a spectral log-segmental SNR in dB.

A preliminary performance comparison of the proposed algorithm with the spectral subtraction algorithm, indicated that with the proposed system the enhanced speech is free of the "musical noise" characteristic to the spectral subtraction, while, aside of the different nature of the residual noise, the enhanced speech sounds approximately the same. A quantitative comparison of these two algorithms will be performed after obtaining an optimal estimator for the a-priori SNR γ_A. This problem is now under study. Nevertheless, we remind the reader that it is possible in the proposed system to obtain the performance of the spectral subtraction algorithm, by letting $\alpha=0$ in (11).

IV. SUMMARY AND CONCLUSIONS

A speech enhancement system which utilizes an optimal m.m.s.e. short-time spectral amplitude estimator is described. The derivation of the optimal estimator is based on modeling speech as a quasi-periodic signal and on applying spectral decomposition. The optimal spectral amplitude estimator coincides with the Wiener spectral amplitude estimator at high SNR values, and is found to be superior to it at low SNR values. It was also noted that the optimal spectral amplitude estimator and a recently developed vector spectral subtraction amplitude estimator [1], are nearly equivalent.

The enhanced speech obtained by using the proposed system suffers less spectral distortion, although contains some more residual noise, than the enhanced speech obtained by using the Wiener spectral amplitude estimator, in the same system. In addition, it is free of the "musical noise" characteristic to the spectral subtraction algorithm.

We believe that the potential of the optimal spectral amplitude estimator proposed here, was not yet fully exploited in this work, since better results can certainly be obtained if the estimation of the a-priori SNR will be improved. This key issue is now being investigated.

APPENDIX A.

In this appendix we derive the optimal amplitude estimator given by (7). Substituting (5) and (6) into (4), we get:

$$\hat{A}_{opt} = \frac{\int_0^\infty a^2 exp\left(-\frac{a^2}{2\sigma^2}\right)\int_{-\Omega}^{\Omega}\frac{1}{2\Omega}\int_0^{2\pi} exp\left(\frac{a r_n}{\sigma_d^2}\cos\Phi_n(\varphi)\right)\frac{d\varphi}{2\pi}d\,\omega_\Delta da}{\int_0^\infty a\,exp\left(-\frac{a^2}{2\sigma^2}\right)\int_{-\Omega}^{\Omega}\frac{1}{2\Omega}\int_0^{2\pi} exp\left(\frac{a r_n}{\sigma_d^2}\cos\Phi_n(\varphi)\right)\frac{d\varphi}{2\pi}d\,\omega_\Delta da} \tag{A.1}$$

where,

$$\Phi_n(\varphi) \stackrel{\Delta}{=} \vartheta_n mod\,2\pi - (\omega_\Delta n+\varphi) mod\,2\pi \tag{A.2}$$

$$\frac{1}{\sigma^2} \stackrel{\Delta}{=} \frac{1}{\sigma_A^2} + \frac{1}{\sigma_d^2} \tag{A.3}$$

The inner integral in (A.1) equals the modified Bessel function of zero order, $I_0(a r_n/\sigma_d^2)$, and is independent of ω_Δ. Therefore, $\hat{A}_{opt}$ is given by:

$$\hat{A}_{opt} = \frac{\int_0^\infty a^2 exp\left(-\frac{a^2}{2\sigma^2}\right)I_0\left(\frac{a r_n}{\sigma_d^2}\right)da}{\int_0^\infty a\,exp\left(-\frac{a^2}{2\sigma^2}\right)I_0\left(\frac{a r_n}{\sigma_d^2}\right)da} \tag{A.4}$$

By defining $u_n \stackrel{\Delta}{=} r_n\sigma^2/\sigma_d^2$, and multiplying both numerator and denominator of (A.4) by $exp(-u_n^2/2\sigma^2)/\sigma^2$, we get:

$$\hat{A}_{opt} = \frac{\int_0^\infty \frac{a^2}{\sigma^2} exp\left(-\frac{a^2+u_n^2}{2\sigma^2}\right)I_0\left(\frac{a u_n}{\sigma^2}\right)da}{\int_0^\infty \frac{a}{\sigma^2} exp\left(-\frac{a^2+u_n^2}{2\sigma^2}\right)I_0\left(\frac{a u_n}{\sigma^2}\right)da} \tag{A.5}$$

The integrand in the denominator of (A.5) is the PDF of a Rician distributed r.v. with parameters (u_n^2, σ^2). Therefore, the denominator of (A.5) equals one, and the numerator of (A.5) equals the expected value of that r.v.. The k-th moment of such a Rician r.v. is given by [5]: $(2\sigma^2)^{k/2}\Gamma(1+k/2)M(-k/2;\,1;\,-u_n^2/2\sigma^2)$, where $M(\alpha;\beta;x)$ is the confluent hypergeometric function. Using this formula with k=1, and γ_A and γ_n which were defined in Section II, we get:

$$\hat{A}_{opt} = \Gamma(1.5)\sqrt{\frac{\gamma_A}{1+\gamma_A}\,\frac{1}{\gamma_n}}\,M\left(-0.5;1;-\frac{\gamma_A}{1+\gamma_A}\gamma_n\right)R_n \tag{A.6}$$

$\hat{A}_{opt}$ as given by (7) is finally obtained from (A.6) by using the following relationship [5]:

$$M(-0.5;1;-x) = exp(-x/2)[(1+x)I_0(x/2)+xI_1(x/2)] \tag{A.7}$$

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