STATISTICAL DESIGN OF ANALYSIS/SYNTHESIS SYSTEMS WITH QUANTIZATION

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ABSTRACT

A statistical model is used for the optimal design of analysis/synthesis systems which include quantization of the signals in the separate bands. Two error measures are used. One is a generalization of the usual statistical mean square error (MSE) to time-varying systems (since analysis/synthesis systems with decimation and interpolation are time varying). The second measure is the time average of the expected ℓ_2 distance between the output of the analysis stage and the analyzed reconstructed signal. The proposed design methods are based on minimizing these error measures and shown to apply not only with the DFT but also with any linear regular transform (e.g. Hadamard, DCT). The above two error measures are shown to be equivalent for a wide class of transforms (including the DFT). The design methods is applicable to either finding an optimal synthesis window for a given analysis window, or finding an optimal analysis window for a given synthesis window. The optimal windows (filters) are obtained by solving a set of linear equations. An optimal analysis/synthesis system is obtained using an iterative algorithm which is based on alternately solving these two sets of linear equations. When no quantization is applied the new design methods coincides with previously reported methods.

I. INTRODUCTION

Analysis/synthesis systems are widely used in speech processing [2-6]. A typical application is medium rate waveform coding (e.g. [2]), where quantization is applied to the signals in separate bands.

Known methods for the design of analysis/synthesis windows (interpreted as analysis/synthesis filter banks [2]), are based on deterministic error measures [3-5], and ignore the quantization noise. Thus, in the presence of quantization noise even a unity system is not necessarily an optimal one, and its performance depends also on the input statistics A statistical approach which will take into account the quantization noise in the design process is therefore needed. Such an approach is presented here, where the statistical analysis is done under the assumption that the quantization noise is additive and is independent of the input signal. This assumption is reasonable if the output SNR is sufficiently high (e.g. for waveform coding schemes at 16Kbps rate and above).

Analysis/synthesis systems are usually used in conjunction with the DFT transform. However, the statistical approach is applied here to a more general model in which any linear regular transform may be used within a fixed time-reference analysis/synthesis system [1].

The model details are as follows: The input signal sequence $x(n)$ is multiplied by the sliding analysis window $h(\cdot)$ having a length of L_h samples. A time aliasing operation reduces these L_h values into a vector of length M (the transform size), denoted by $\underline{x}_s$. These vectors (for different time instances) are decimated with a decimation factor R, $1 \leqslant R \leqslant M$. The m-th element of the vector $\underline{x}_{sR}$ is given by:

$$x_{sR}(m) = \sum_{r=-\infty}^{\infty} h(sR-m-Mr)\cdot x(m+Mr), \quad 0 \leqslant m \leqslant (M-1) \quad (1)$$

A linear regular transform of size M operates on these vectors and results in output vectors $\underline{X}_{sR}$ of M elements each. This transform is represented by the square matrix T of dimensions $M \times M$, whose (k,m) element is denoted by $t(k,m)$, $0 \leqslant k,m \leqslant (M-1)$ (e.g., for the DFT, $t(k,m) \triangleq \exp\left(-j\,\frac{2\pi km}{M}\right)$). The k-th element of the vector $\underline{X}_{sR}$ is given by:

$$X_{sR}(k) = \sum_{m=0}^{M-1} t(k,m)x_{sR}(m), \quad 0 \leqslant k \leqslant (M-1) \quad (2)$$

This completes the analysis part of the system. If the DFT transform is applied, the output of the analysis stage is known as the discrete short-time Fourier transform (DSTFT) [5], of the input signal. For a general transform we denote the output of the analysis stage as the discrete short-time transform (DSTT), of the input signal. The quantization is modeled by an additive noise vector $\underline{V}_{sR}$ which is added to the output signal from the analysis stage. The k-th element of the modified discrete short-time transform (MDSTT) vector $\hat{\underline{X}}_{sR}$ is given therefore by:

$$\hat{X}_{sR}(k) = X_{sR}(k) + V_{sR}(k), \quad 0 \leqslant k \leqslant (M-1) \quad (3)$$

The MDSTT is the input of the systhesis stage. An inverse transform operates on the MDSTT and results in time domain vectors $\hat{\underline{x}}_{sR}$ of M elements each. The inverse transform is represented by the matrix T^{-1} of dimensions $M \times M$, whose (m,k) element is denoted by $t^{-1}(m,k)$ for $0 \leqslant m,k \leqslant (M-1)$. The m-th element of the vector $\underline{x}_{sR}$ is given by:

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$$\hat{x}_{sR}(m) = \sum_{k=0}^{M-1} t^{-1}(m,k)\hat{X}_{sR}(k), \quad 0 \leq m \leq (M-1) \qquad (4)$$

The sequence of time domain vectors x_{sR} is interpolated with an interpolation factor $\bar{R}$. A wieghted overlapp-add operation reconstruct the output sequence $y(n)$ from these vectors. This operation is done using a synthesis window $f(\cdot)$ of L_f samples, and is described by:

$$y(n) = \sum_{s=-\infty}^{\infty} f(n-sR)\hat{x}_{sR}((n)_M) \qquad (5)$$

where $(n)_M$ denotes $n \mod M$.

Since both the analysis and synthesis windows are of finite length the summations in (1) and (5) are actually finite. An infinite summation range is used here to simplify the presentation, and for the same reason will also be used in the sequel.

Let v_{sR} be the inverse transform of the noise vector $\underline{V}_{sR}$. Thus the m-th element of $\underline{v}_{sR}$ is given by:

$$v_{sR}(m) = \sum_{k=0}^{M-1} t^{-1}(m,k)V_{sR}(k), \; 0 \leq m \leq (M-1) \qquad (6)$$

If the transform is complex (e.g. a DFT), we assume that the quantization of the DSTT vectors is done in a way that assures the realness of $\underline{v}_{sR}$. This is needed in order to obtain a real output sequence.

We assume that the input signal and the elements of the random vectors $\underline{v}_{sR}$ are samples of wide-sense stationary processes with zero-mean. Furthermore, we assume that the elements of the random vectors v_{sR} are un-correlated with the input signal. Written formally:

$$E[x(n)] = E[v_{sR}(m)] = 0 \quad \forall n,s;\; 0 \leq m \leq (M-1) \qquad (7)$$

$$E[x(n) \cdot v_{sR}(m)] = 0 \quad \forall n,s;\; 0 \leq m \leq (M-1) \qquad (8)$$

$$E[x(n) \cdot x(n+d)] \triangleq \phi^{x}(d) \quad \forall n,d \qquad (9)$$

$$E[v_{sR}(m) \cdot v_{(s+d)R}(n)] \triangleq \phi^{v}_{m,n}(R \cdot d) \quad \forall s,d;\; 0 \leq m,n \leq (M-1) \qquad (10)$$

$\phi^{x}(\cdot)$ is the autocorrelation sequence of the input signal, and $\phi^{v}_{m,n}(R\cdot d)$ is the (m,n) element of the autocorrelation matrix at lag $R\cdot d$ of the noise vectors. If the input is a speech signal, $\phi^{x}(\cdot)$ represents the long-term autocorrelation sequence of speech, thus incorporating the non-flatness of the speech spectrum in the design process. Equations (1)-(10) completely define the model on which the statistical approach is based.

In the next section, we define two error measures, which are shown to be equivalent for a wide class of transforms. The optimal design is based on minimizing these measures. In Section III various design methods for optimal analysis/synthesis are presented along with the conditions under which the new approach coincides with previously reported results [3-5].

II. ERROR MEASURES

Physically realizable analysis/synthesis systems introduce signal delay. It is known that exact signal reconstruction is possible only if this delay is an integer multiple of the transform size [6]. We, therefore, assume that the analysis/synthesis system has a delay $M\cdot r_0$, with r_0 being an integer. Since in an ideal system the reconstructed signal coincides with the delayed input signal, we define the output error signal as:

$$\varepsilon(n) \triangleq y(n) - x(n-Mr_0) \qquad (11)$$

Since the analysis/synthesis system considered is linear, and both its input and the additive noise have zero mean, the output error signal has also zero-mean. However, since it is a time-varying system (because of embedded decimation and interpolation), $\varepsilon(n)$ is <u>not</u> a wide-sense stationary process. We now derive a generalization of the usual MSE error measure for the non-stationary process $\varepsilon(n)$.

Let the autocorrelation sequence of $\varepsilon(n)$ be denoted by $\phi^{\varepsilon}(\cdot)$, i.e.:

$$\phi^{\varepsilon}(d,m) \triangleq E[\varepsilon(m+d)\cdot\varepsilon(m)] \quad \forall d,m \qquad (12)$$

An expression for $\phi^{\varepsilon}(\cdot)$ is given in the Appendix. For any analysis/synthesis system, this function has the following two properties:

$$|\phi^{\varepsilon}(d,m)| \leq C \quad \forall d,m \qquad (13a)$$

$$\phi^{\varepsilon}(d,m+\ell\cdot p) = \phi^{\varepsilon}(d,m) \quad \forall d,m,\ell \qquad (13b)$$

where $p \triangleq R\cdot M/\gcd(r,M)$ ($\gcd(R,M)$ is the greatest common divisor of the two integer numbers R and M). The derivation of these two properties is also given in the Appendix.

Let $G(f)$ be a non-negative real and symmetric weight function, whose Fourier coefficients, $\phi^{g}(n) \triangleq \int_{-0.5}^{0.5} G(f)e^{j2\pi fn}df$ converges absolutely, i.e.:

$$\lim_{\ell\to\infty} \sum_{n=0}^{\ell} |\phi^{g}(n)| < \infty \qquad (14)$$

The first error measure considered is defined by:

$$U \triangleq \sum_{d=-\infty}^{\infty} \phi^{g}(d)\cdot\left[\frac{1}{p}\sum_{m=0}^{p-1}\phi^{\varepsilon}(d,m)\right] \qquad (15)$$

Due to (13a) and (14), this error measure is bounded. The following theorem (presented here without proof because of lack of space), gives a spectral domain interpretation of this error measure:

<u>Theorem 1:</u>

Let:

$$u_{\ell,r} \triangleq \frac{1}{(2r+1)} \int_{-0.5}^{0.5} E\left|\sum_{n=\ell-r}^{\ell+r} \varepsilon(n)e^{-j2\pi fn}\right|^2 G(f)df \qquad (16)$$
$$\forall \ell, r$$

(a) For every value of ℓ, the sequence $u_{\ell,r}$ converges to U as $r \to \infty$. Therefore U is non-negative.

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(b) For $\varepsilon(n)$ that is a wide-sense stationary process, having a spectrum $S(f)$, it follows that:

$$U = \int_{-0.5}^{0.5} S(f)G(f)df \ . \tag{17}$$

The error signal $\varepsilon(n)$ was defined in the time-domain. An alternative approach is to define the error in the transform domain. Let $\underline{Y}_{sR}$ be the DSTT of the reconstructed signal $y(n)$. In an ideal system, this vector is a delayed version of the DSTT of the input signal (with delay $M\cdot r_o$). The error vector in the transform domain is defined therefore as:

$$\underline{\varepsilon}_{sR} \triangleq \underline{Y}_{sR} - \underline{X}_{(sR-Mr_o)} \ . \tag{18}$$

The DSTT is a linear operation, therefore $\underline{\varepsilon}_{sR}$ has zero mean. Let $D(sR)$ be the expected value of ℓ_2 norm of the random error vector $\underline{\varepsilon}_{sR}$, i.e.:

$$D(sR) \triangleq E[\|\underline{\varepsilon}_{sR}\|^2] = \sum_{k=0}^{M-1} E[|Y_{sR}(k)-X_{sR-Mr_o}(k)|^2] \tag{19}$$

It follows from the definition of the DSTT ((1) and (2)), and of $\varepsilon(n)$ (in (11)), that $D(sR)$ can be represented in terms of the autocorrelation sequence $\phi^{\varepsilon}(\cdot)$ as:

$$D(sR) = \sum_{r=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} h(sR-t)h(sR-r)\phi^{\varepsilon}(r-t,t)\cdot a((r)_M,(t)_M) \tag{20}$$

where $a(i,j)$ is the (i,j) element of the $M\times M$ matrix A defined as:

$$A = T^*\cdot T \tag{21}$$

where * denotes 'conjugate transpose'. The error vector $\underline{\varepsilon}_{sR}$ is not a wide-sense stationary process because the analysis/synthesis system is time-varying. Therefore, the value of $D(sR)$ depends on the time instances sR. However, due to property (13b), this function is periodic with a period of p, i.e.:

$$D(sR+\ell\cdot p) = D(sR) \qquad \forall s,\ell \ \text{integers} \tag{22}$$

We now define the second error measure as the time average of $D(sR)$ over one period of this periodic function, i.e.:

$$V \triangleq \frac{1}{p}\sum_{m=0}^{p-1} D(sR+m) \tag{23}$$

Substituting (20) in (23) results in the following equation:

$$V = \sum_{d=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} h(n)h(n+d)\right]\cdot\frac{1}{p}\sum_{m=0}^{p-1}[\phi^{\varepsilon}(d,m) \ a((m+d)_M,(m)_M)] \tag{24}$$

Comparing (24) with (15) it is easily verified that when the matrix A is a circulant matrix (i.e., $a(i,j) = a((i+d)_M,(j+d)_M)$ for $d=0,\ldots,M-1$), U and V are equivalent measures. In this case, if the values of $\phi^g(\cdot)$ are set according to:

$$\phi^g(d) \triangleq \sum_{n=-\infty}^{\infty} h(n)\cdot h(n+d)\cdot a((d)_M,0) \tag{25}$$

Then U and V become equal, independent of the synthesis filter and decimation factor used. Both U and V measure the quality of the signal reconstruction for a given variance of the quantization noise. In typical applications it is also desired that the analysis window will have a low-pass frequency response. Therefore, both U and V should be modified in order to incorporate this specification in the design process. Following the approach in [3], we add to the error measure the weighted mean square error between the frequency response of the analysis window $H(f)$ and the desired frequency response $D(f)$ of the LPF prototype. Let $W(f)$ denote the weight function, and $\hat{U}$ and $\hat{V}$ the modified error measures, then:

$$\hat{U} = U + \int_{-0.5}^{0.5} W(f)|H(f) - D(f)|^2 df \tag{26a}$$

$$\hat{V} = V + \int_{-0.5}^{0.5} W(f)|H(f) - D(f)|^2 df \tag{26b}$$

III. DESIGN OF OPTIMAL SYSTEMS

We consider three different types of design problems:

(a) The design of an optimal synthesis window for a given analysis window.
(b) The design of an optimal analysis window for a given synthesis window.
(c) The design of an optimal analysis/synthesis system.

We consider $\hat{U}$ as the error measure, and the optimality criterion is the minimization of $\hat{U}$ with respect to the unknown window or windows. For a wide class of transforms (including the DFT) for which the matrix A in (21) is circulant, the two error measures $\hat{U}$ and $\hat{V}$ are equivalent, and therefore similar results can be obtained when $\hat{V}$ is used as the error measure.

When the analysis window is given, $\hat{U}$ in (26a) can be written as a p.s.d. quadratic form in terms of the synthesis window $\underline{f}$, i.e.:

$$\hat{U} = C_h + \frac{1}{R}(\underline{f}^T\cdot(Q_h+Q_v)\cdot\underline{f} - \underline{b}_h^T\cdot\underline{f}) \tag{27}$$

The expressions for the matrices Q_h and Q_v, which are of dimensions $L_f\times L_f$, the error vector $\underline{b}_h$ which is of length L_f and the constant C_h, are given in the Appendix. The solution of problem (a) above, is thus obtained by solving the following set of linear equations:

$$(Q_h + Q_h^T + Q_v + Q_v^T)\cdot\underline{f}_{opt} = \underline{b}_h \tag{28}$$

If this set of equations is degenerate, any-one of the infinitely many possible solutions have the same (minimal) value of $\hat{U}$. The matrix Q_v reflects the quantization noise effect, and Q_h depends on the analysis - window sequence. It can be verified that if no quantization is applied ($\phi^v(\cdot)=0$), and a unity system exist (which requires $R<M$ or $L_f = L_h = M$), any unity system is a solution of the (possibly degenerate) set of equations (28), regardless of the weight function $G(f)$ and input signal statistics. Thus for this case the result presented in [4,5] is but one of many possible optimal solutions of problem (a). When $R=M$ and $L_f,L_h>M$ (which is the typical situation in waveform coding applications), no unity system exists [6], and the set of equations in (28) provides a unique solution, even if no quantization is applied.

For this case ($R=M$, $L_f, L_h > M$) the statistical approach not only accounts for the quantization noise but also utilizes the statistical properties of the input signal.

In order to illustrate the effect of the quantization noise, we consider now a simple example for which an analytic solution of (28) can be obtained. We assume that the input signal has a flat spectrum and its variance is σ_x^2, and that the elements of the noise vector $V_{SR}(k)$ are uncorrelated random variables with equal variance denoted by σ_v^2. We now use the error measure $\hat{U}$ with a uniform spectral weight ($G(f) = 1$). For the special case of $L_f=L_h=M$, $r_0=1$, and a symmetric analysis window, the optimal synthesis filter is:

$$f_{opt}(t) = \frac{h(t)}{\sum_{m=-\infty}^{\infty} h(t-mR)^2 + (\sigma_v^2/\sigma_x^2)} \quad 0 \leq t \leq (M-1) \tag{29}$$

which extends the results in [4,5], to include the effect of the quantization noise.

We consider now the solution of problem (b), i.e. the design of an optimal analysis window given a systhesis window. Again $\hat{U}$ can be written as a p.s.d. quadratic form in terms of the analysis window $\underline{h}$, i.e.:

$$\hat{U} = C_f + \frac{1}{R}\left(\underline{h}^T(Q_f + Q_D)\cdot\underline{h} - (\underline{b}_f+\underline{b}_D)^T\cdot\underline{h}\right) \tag{30}$$

The expressions for the matrices Q_f and Q_D which are of dimensions $L_h \times L_h$, the vectors $\underline{b}_f$ and $\underline{b}_D$ of length L_f each, and the constant C_f, are given in the Appendix. The solution of problem (b) is thus obtained by solving the following set of linear equations:

$$(Q_f + Q_f^T + Q_D + Q_D^T)\cdot\underline{h}_{opt} = (\underline{b}_f + \underline{b}_D) \tag{31}$$

Again, if the set of equations is degenerate, anyone of the possible solutions have the same (minimal) value of $\hat{U}$. However, once $W(f)$ is positive on a set of non-zero measure, there is a unique solution of (31), which is a compromise between the desired LPF specification, and the unity system specification for the given synthesis window. The matrix Q_D and the vector $\underline{b}_D$ reflect the specifications on the desired LPF frequency response of the analysis window, whereas Q_f and $\underline{b}_f$ which depend on the synthesis window reflect the unity system specification.

The design of an optimal analysis/synthesis system (problem (c) above), is done by an iterative algorithm, consisting of alternately solving the above two sets of linear equations ((28) and (31)), similar to [3].

We consider now the design of an optimal analysis/synthesis system for the special case which satisfies the following conditions: (i) the decimation factor equals the transform size ($R=M$); (ii) the input signal is white; (iii) a uniform frequency weighting of the error is used; (iv) no quantization is applied.

For this case, $\hat{U}$ is similar to the error measure in [3] which is a deterministic MSE between the analysis/synthesis system's impulse response and the ideal response. The only difference is that we incorporated the linear constraint on the gain of the system, used in [3] into the error measure, thus avoiding the need for using Lagrange multipliers and hence simplifying the design algorithm.

IV. CONCLUSIONS

A statistical model of analysis/synthesis with quantization was presented. This model can be applied with the DFT or any other linear regular transform. Under the presented statistical model, two different error measures were defined and shown to be equivalent for a wide class of transforms (including the DFT). The optimal synthesis window for a given analysis window, and the optimal analysis window for a given synthesis window are obtained by solving a set of linear equations. An iterative algorithm (similar to [3]), with global convergence properties is used to design an optimal analysis/synthesis system. For $R < M$, and no-quantization noise, the result in [4,5] is one of many possible optimal synthesis windows according to the statistical approach presented here. However, for a white input signal, white quantization noise and uniform frequency weighting, the optimal synthesis window obtained here extends the result in [4,5] to include the effect of quantization noise. For $R = M$, the statistical approach partially extends the results of [3], by allowing non-zero quantization noise, and incorporating into the design process the input signal statistics, and the error frequency weighting. A statistical model for systems with QMF filters [3], and quantization noise is now studied.

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