DESIGN OF FILTER BANKS WITH SPECIFIED COMPOSITE RESPONSE AND MAXIMUM OUTPUT SNR

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Abstract

A new method for designing uniform and non-uniform digital filter banks with specified composite response is presented. The individual filters in the bank are FIR (finite impulse response) digital filters with linear phase.

The new method maximizes the weighted harmonic mean of the output signal to noise ratio (SNR) of the individual filters and guarantees the fulfillment of the specified composite response. The method is shown to be equivalent to an optimal weighted minimum mean square error design (WMMSE) with specified composite response and allows the design of filter banks with variable individual filters lengths. For uniform filter banks the method is simplified and is applied directly to the design of a lowpass prototype. The new method is demonstrated by a design example.

I. Introduction

In many applications filter banks with specified composite response (usually flat), are required. We shall focus on filter banks in which the individual filters are FIR (finite impulse response) digital filters with linear phase, and real coefficients.

Direct application of the well known Remez exchange method [1], for the design of filter banks, generally results in a poor composite response [2]. This is particularly so for non-uniform filter banks where even a complicated automated trial and error approach of iterated designs using the Remez exchange method might diverge [3].

Various methods exist that guarantee a flat composite response [2,4,5,6]. However they all suffer from the following disadvantages:

(a). Sub-optimality under both Min-Max and WMMSE criteria.

(b). Limited flexibility in the design, (e.g., restriction to individual filers of equal length).

The new method presented in this work overcomes these disadvantages, and will be illustrated via a design example. The filter design criterion of maximum output SNR is not new and was already used for the design of a single bandpass/bandstop FIR filter with linear phase [7]. Its connection with the WMMSE criterion for this problem was presented in [8].

However, these results are not applicable to the design of filter banks with specified composite response, since there is no control on the resulting composite response. In the new method presented here the composite response is incorporated into the design process as a constraint on the sum of the individual filters, and an overall weighted mean square error of these filters is minimized subject to that constraint. A set of weighting constants reflects the importance of each individual filter.

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The general mathematical framework of the new method is presented in [2]. In this work we present the new method from a statistical point of view. This interpretation results (similarly to [7]) in a relation between relative input signal and noise levels, and the weight function used in the WMMSE design. Thus the resulting filter bank maximizes the weighted harmonic mean of the output SNR's, subject to the specified composite response constraint. In the next section the statistical approach for presenting the new method is given for the general case of non-uniform filter banks. The mathematical derivations as well as the relation between this approach and the WMMSE criterion are given in Appendix A. Section III deals with uniform filter banks. For these filter banks the *optimal* individual filters are shown to be frequency translated versions of a lowpass prototype. Thus, a simplified version of the new method is derived and can be applied directly to the design of the lowpass prototype. The details of the mathematical derivations are given in Appendix B. The last section demonstrates the new method via a design example and conclusions are drawn.

II. Filter bank design using a Maximum output SNR criterion

We shall state the design problem from a statistical point of view for filter banks with real input and real coefficients. Generalization of the method for complex signals and filters is not difficult.

Since each filter in the filter bank is usually designed to pass a different frequency band of the common input signal, we may define differently the so called signal and noise components for each filter in the bank. The convention taken here is to consider all the frequency components of the common input which are in the pass band of the i-th filter as its input signal s_i and all the components in the stopband as the noise n_i. Because we deal with a filter design problem, components in the transition bands of each individual filter are ignored. Thus we view each filter as having its own separate input denoted by $x_i = s_i + n_i$, for the i-th filter in the bank. Note that according to the above convention the inputs x_i, $i=1,2,..,N$ (for N filters in the filter bank) will all be identical if all the transition bands are eliminated (i.e. set to have a zero bandwidth). For the mathematical development to follow it is convenient to apply the following vector notation:

The impulse response of the i-th filter, which is of length M_i, is denoted by the vector $\underline{a}_i$ (i.e., $\underline{a}_i \in I\!R^{M_i}$). The input vector, which comprises of M_i consecutive samples of the random process x_i, is denoted by $\underline{X}_i(k)$, i.e. $\underline{X}_i(k)=[x_i(k), x_i(k-1),..,x_i(k-(M_i-1))]^T$. Thus the corresponding output is $y_i(k)=\underline{a}_i^T \cdot \underline{X}_i(k)$. As explained above we regard input samples as being the sum of signal samples and noise samples, and we assume that they are samples of two uncorrelated, zero mean, wide-sense stationary continuous random processes. In vector notation we have $\underline{X}_i(k)=\underline{S}_i(k)+\underline{N}_i(k)$ where $\underline{S}_i(k)$ contain signal samples and $\underline{N}_i(k)$ contain noise samples, in the i-th filter input. The *desired* signal at the i-th output at time k is defined to be the delayed version of the input signal, i.e., $y_i^d(k)=s_i(k-d_i)=\underline{u}_{d_i}^T\cdot\underline{S}_i(k)$, where $\underline{u}_{d_i}$ is the d_i-th unit vector in $I\!R^{M_i}$, i.e. all its elements are zero except the d_i-th element which is one. We now divert from the usual convention of assuming the signal component at the output as the response of the filter to the signal component at the input and we set the signal component at the output of the i-th filter to be $y_i^d(k)$, which is independent of the filter $\underline{a}_i$. This way, the noise component at the output of the i-th filter contains both the filtered input noise, and the distortions of the input signal introduced by the i-th filter. With these assumptions the signal power at the output of the i-th filter is given by:

$$S_{o_i} = E[y_i^d(k)^2] = \underline{u}_{d_i}^T\cdot R_{ss_i}\cdot\underline{u}_{d_i} = r_{ss_i}(o) \tag{1}$$

and the corresponding noise power is:

$$N_{o_i}=E[(y_i(k)-y_i^d(k))^2]=r_{ss_i}(o)-2\cdot\underline{a}_i^T\cdot R_{ss_i}\cdot\underline{u}_{d_i}+\underline{a}_i^T\cdot R_{xx_i}\cdot\underline{a}_i \tag{2}$$

where R_{ss_i}, R_{xx_i}, and R_{nn_i} are $M_i\times M_i$ autocorrelation matrices defined by:

$$R_{xx_i} = E[\underline{X}_i(k)\cdot\underline{X}_i(k)^T] = R_{ss_i}+R_{nn_i} \tag{3}$$

$$R_{ss_i} = E[\underline{S}_i(k)\cdot\underline{S}_i(k)^T] \quad ; \quad R_{nn_i} = E[\underline{N}_i(k)\cdot\underline{N}_i(k)^T]$$

with $r_{ss_i}(k)$ and $r_{nn_i}(k)$ being the corresponding autocorrelation sequences. Maximizing the output SNR of each of the individual filters leads to the well known Wiener filter solution [7]:

$$\underline{a}_i^{\bullet} = R_{xx_i}^{-1}\cdot R_{ss_i}\cdot\underline{u}_{d_i} \quad \text{for} \quad i=1,..,N. \tag{4}$$

With the corresponding output SNR (denoted by B_i):

$$B_i = \frac{r_{ss_i}(o)}{r_{ss_i}(o)-\underline{u}_{d_i}^T\cdot R_{ss_i}\cdot R_{xx_i}^{-1}\cdot R_{ss_i}\cdot\underline{u}_{d_i}} \tag{5}$$

Independent designs of the individual filters, using (4), may result however in a poor composite response. To solve this problem a composite response specification is now incorporated into the design process. The desired composite response is specified as the frequency response of a desired FIR composite filter $\underline{a}_c^{\bullet}$ (e.g., for flat composite response $\underline{a}_c^{\bullet}$ is a unit vector). Since in general each individual filter may be specified to have a different delay with respect to its input signal, the composite response is defined as the sum of the frequency responses of the properly delayed individual filters, such that all the delayed individual filters have the same delay. This adjusted common delay is denoted by d_c. The desired composite response filter is of length M_c which satisfies $M_c=Max\{M_i, i=1,..,N\}$. Thus the composite response of the designed filter bank satisfies the desired specification if and only if:

$$\sum_{i=1}^{N}\underline{a}_i^{aug} = \underline{a}_c^{\bullet} \tag{6}$$

Where $\underline{a}_i^{aug}\in I\!R^{M_c}$, is an augmented version of $\underline{a}_i\in I\!R^{M_i}$, obtained by padding zeros (on both ends of $\underline{a}_i$ - as necessary to adjust the filter's delay), so that the d_i element of $\underline{a}_i$ becomes the d_c element of $\underline{a}_i^{aug}$.

We shall denote by S_c the set of all filter banks that satisfy this constraint (i.e., $S_c=\{\{\underline{a}_i\}_{i=1}^N | \sum_{i=1}^{N}\underline{a}_i^{aug}=\underline{a}_c^{\bullet}\}$). Of course, the specification of $\underline{a}_c^{\bullet}$ should match the specifications of the individual filters, for example, if a flat composite response is desired, $\underline{a}_c^{\bullet}$ should be *the* d_c unit vector of $I\!R^{M_c}$, where d_c is the adjusted common delay mentioned above.

Because of the constraint that the filter bank must belong to S_c, all N individual filters must be designed simultaneously. As noted earlier, the optimization criterion which we use for selecting the best filter bank in the set S_c is maximum weighted harmonic mean of the N output SNR's, i.e.

$$\underset{\{\underline{a}_i\}_{i=1}^N\in S_c}{Max}\left\{\frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{C_i}{(S_{o_i}/N_{o_i})}}\right\} \tag{7a}$$

This is equivalent to minimizing the sum in the denominator, i.e.,

$$\underset{\{\underline{a}_i\}_{i=1}^N\in S_c}{Min}\left\{\sum_{i=1}^{N}C_i\cdot\frac{N_{o_i}}{S_{o_i}}\right\} \tag{7b}$$

$C_i>0$ are relative weight constants which are proportional to the desired output SNR of the corresponding filters.

The harmonic mean is dominated by the lower SNR values, thus providing an approximation to a Max-Min criterion. This is in contrast to the arithmetic mean (which may appear at first more natural), which is dominated by the higher SNR values and may lead to few filters having very high output SNR, and others having very low output SNR.

Note that if the filter bank $\{\underline{a}_i^{\bullet}\}_{i=1}^N$, obtained from the Wiener solution (4) (to be denoted the "Wiener filter bank") also satisfies the desired composite response constraint (i.e. it is in S_c), it is obviously also the optimal filter bank, i.e. the solution of (7). However if the Wiener filter bank is not in S_c, each filter in the optimal filter bank (which is in S_c) cannot have a higher value of output SNR than the corresponding filter in the Wiener filter bank (which does not satisfy the composite response constraint). Thus the composite response constraint is expected to degrade the performance of the individual filters, with this degradation increasing as the corresponding weight constant C_i becomes smaller, relative to other filters' weights. Note also that selecting one of the weights to be large relative to the others will improve the performance of the corresponding filter. However, this will be on account of the performances of the other filters in the bank. Since S_{o_i} does not depend on $\underline{a}_i$ (see (1)), the optimization criterion in (7b) is equivalent to the minimization of the weighted output noise power (which is related to a WMMSE design), subject to constraint (6). The relation with WMMSE design is further elaborated in Appendix A, together with the details of the design method.

The resulting optimal filters are found to be (see Appendix A):

$$\underline{a}_i = \underline{a}_i^{\bullet} + \frac{r_{ss_i}(o)}{C_i}R_{xx_i}^{-1}\cdot\underline{q}\,|_{M_i} \quad , \quad i=1,..,N \tag{8}$$

with the corresponding output SNR's being:

$$\frac{S_{o_i}}{N_{o_i}} = \frac{B_i}{1+\frac{B_i}{C_i^2}r_{ss_i}(o)\cdot\underline{q}\,|_{M_i}^T\cdot R_{xx_i}^{-1}\cdot\underline{q}\,|_{M_i}} \quad , \quad i=1,..,N \tag{9}$$

where $\underline{a}_i^{\bullet}$ and B_i are given in (4) and (5) respectively, and $\underline{q}$ is defined by (A8) in Appendix A. Reducing properly the vector $\underline{q}$ from $I\!R^{M_c}$ into $I\!R^{M_i}$ results in $\underline{q}\,|_{M_i}$. The reduction is done such that the d_c element of $\underline{q}$ becomes the d_i element of $\underline{q}\,|_{M_i}$, with the elements at both ends of the vector $\underline{q}$ being eliminated.

It should be noted that the optimal individual filters have linear phase if:

$$d_i = (M_i-1)/2 \quad \text{for} \quad i=1,..,N \tag{10a}$$

and:

$$\underline{a}_c^{\bullet} \textit{ has linear phase} \tag{10b}$$

The proof of this claim is given in Appendix A.

We shall elaborate now on setting the design parameters in (3). The input signal and noise components of each individual filter are two non-overlapping narrow-band processes with known cut-off frequencies, since, as explained earlier, the signal component is related to the passband and the noise to the stopbands. There is typically also a-priori knowledge on the input SNR (which is inversely proportional to the dynamic range (DR) in the frequency domain of the input, since a signal component in one filter is a noise component in other filters).

A typical specification is:

$$\begin{cases} F\{r_{ss_i}(k)\}=0 \ \textit{outside predefined frequency} \\ \qquad \textit{intervals (passbands)} \\ F\{r_{nn_i}(k)\}=0 \ \textit{outside predefined frequency} \\ \qquad \textit{intervals (stopbands)} \end{cases} \quad (11)$$

where $F\{\cdot\}$ denotes Fourier transformation, and

$$\left.\begin{cases} r_{ss_i}(o)=1 \\ r_{nn_i}(o)=[\textit{Input DR of } i\textit{-th filter}]^{-1} \end{cases}\right\} \text{ for } i=1,..,N \quad (12)$$

A reasonable specification of the other elements of $r_{ss_i}(k)$ and $r_{nn_i}(k)$, is such that the signal and noise spectrums are flat in their bands, while satisfying (11) and (12). This results in the same input characteristics as in [7,8]. However, the design method in [7,8], results in Wiener filters as in (4), which in general do not satisfy the composite response specification (6), and thus the new method can be considered to be an extension of these results.

III. Uniform Filter Banks

For the special case of a uniform filter bank with a flat composite response, all input signals have the same bandwidth. Conventional approaches to the design of such filter banks use a lowpass prototype filter [4,5]. N *complex* individual filters can be obtained by frequency translations, (on the normalized frequency axis, with 1 corresponding to the sampling frequency), by $(i+\Delta)/N$, $i=1,..,N$, $0\le\Delta<1$. A real filter bank consisting of $\frac{N}{2}$ or $(\frac{N}{2}+1)$ filters is obtained from the complex filter bank of N filters by combining the complex individual filters in conjugate pairs. To obtain a real filter bank when using a real lowpass prototype filter, Δ should take the value of 0 or 1/2. This issue is explained later on. Thus, in the conventional approaches, the specified composite response puts a constraint on the lowpass prototype filter [5].

If one applies the new method, presented in the previous section, to the design of a real uniform filter bank, having a flat composite response, he will find, in general, that the individual filters in the resulting optimal filter bank are not frequency translations of a single lowpass prototype filter. This means that a real uniform filter bank which is obtained by the conventional approach of frequency translation of a lowpass prototype filter is usually not optimal under the Maximum output SNR criterion defined in the previous section. It also shows that the general solution in (8) cannot be simplified for this problem. However, if we are satisfied with optimality of the complex individual filters, an optimal complex uniform filter bank with a flat composite response can be obtained also by frequency translation of a lowpass filter prototype, provided that the design problem is specified as follows:

(a). All N complex individual filters are of equal length and have the same desired signal delay (which must have an integer value).

(b). The desired composite response is flat.

(c). The desired output SNR values are the same for all N individual filters, thus they all have the same relative weight.

(d). All N individual filters have the same input signal and noise powers, and the same input signal and noise bandwidths. Furthermore, the input signal and noise spectrums of the individual filters are frequency translated versions of a prototype input signal spectrum and a prototype input noise spectrum, respectively. The frequency translation of the i-th individual filter's input is $(i+\Delta)/N$, and the prototype input is a real random process.

Written formally:

$$M_i=M_c,\ d_i=d_c \quad , \quad i=1,..,N\ ; \ d_c \ \textit{an integer.} \quad (13a)$$

$$\underline{a}_c^{\bullet} = \underline{u}_{d_c} \quad (13b)$$

$$C_i = 1 \quad , \quad i=1,..,N \quad (13c)$$

$$\begin{cases} R_{xx_i} = \Delta^{-1}\cdot W^{-i}\cdot R_{xx_0}\cdot W^{i}\cdot\Delta \\ R_{ss_i} = \Delta^{-1}\cdot W^{-i}\cdot R_{ss_0}\cdot W^{i}\cdot\Delta \end{cases} \quad \text{for } i=1,..,N \quad \mathbf{(13d)}$$

where

$$W = diag\{\omega_N,\omega_N^2,...,\omega_N^{M_c}\} \ , \quad \omega_N=e^{j\frac{2\pi}{N}}$$

and:

$$\Delta = diag\{\omega_N^{\Delta},\omega_N^{2\Delta},\cdots,\omega_N^{M_c\Delta}\}$$

and R_{xx_0} and R_{ss_0} are both real toeplitz p.d. matrices.

Under these specifications it is proven in Appendix B that the optimal filter bank resulting from (8), has the following property:

$$\underline{a}_i = \omega_N^{(i+\Delta)d_c}\cdot\Delta^{-1}\cdot W^{-i}\cdot\underline{a}_0 \quad \text{for } i=1,..,N \quad (14)$$

Thus the individual filters are frequency translated versions of the prototype filter $\underline{a}_0$. It is also shown in Appendix B, that this prototype filter has real coefficients, and is found to be given by:

$$\underline{a}_0 = R_{xx_0}^{-1}[R_{ss_0}\cdot\underline{u}_{d_c}+r_{ss_0}(o)\cdot\underline{\hat{q}}] \quad (15)$$

where:

$$\underline{\hat{q}} = \frac{1}{r_{ss_0}(o)}H^T\cdot(H\cdot R_{xx_0}^{-1}\cdot H^T)^{-1}\cdot H\cdot[\frac{1}{N}I-R_{xx_0}^{-1}\cdot R_{ss_0}]\cdot\underline{u}_{d_c} \quad (16)$$

The matrix H in (16) is of dimensions $Q\times M_c$ where $Q=\lfloor\frac{M_c}{N}\rfloor+1$. Each of the Q rows of this matrix is a different unit vector of $\mathbb{R}^{M_c}$, and the Q non-zero columns, are those whose indeces are congruent with d_c modulu N. Equations (14), (15) and (16), thus yield a simplified version of the new method, applied directly to the design of the lowpass prototype in uniform filter banks.

It is also shown in Appendix B, that for this case all N output SNR's of the complex individual filters are the same and their value is:

$$\frac{S_{o_i}}{N_{o_i}} = \frac{B_o}{1+B_o\cdot r_{ss_0}(o)\cdot\underline{\hat{q}}^T\cdot R_{xx_0}^{-1}\cdot\underline{\hat{q}}} \quad (17)$$

where B_o is obtained by substituting the prototype specifications, R_{xx_0} and R_{ss_0}, in (5), with $i=0$.

For $d_c=(M_c-1)/2$ (and of course M_c is odd), (13a) and (13b) are equivalent to (10a) and (10b), respectively, and thus all N individual filters, as well as the prototype filter, have linear phase. This can also be verified directly in the solution's equations (14) - (16). It is also easily verified that for $\Delta=0$, $(\underline{a}_i+\underline{a}_{N-i})$ $i=1,..,N-1$, are real vectors and so is $\underline{a}_N$, thus the complex filter bank can then be reduced to a *real* filter bank of $(N/2)+1$ individual filters. For

$\Delta=1/2$, $(\underline{a}_i+\underline{a}_{N-1-i})$,$i=1,..,N-2$, are real vectors and so is $(\underline{a}_{N-1}+\underline{a}_N)$, thus the complex filter bank can then be reduced to a *real* filter bank of $N/2$ individual filters.

IV. Design Example and Conclusions

To illustrate the new method, the following design example is presented:

The problem we consider is the design of an Octave-band filter bank composed of five individual filters with real coefficients and linear phase. The composite response is specified to be flat, with a composite delay (d_c), equal to the delay of the longest individual filter in the bank. The first filter in the bank is a lowpass filter and the last one is a highpass filter and the other three are bandpass filters. The i-th filter has bandwidth which is twice the bandwidth of the $(i-1)$-th filter, starting with a lowpass filter having a passband width of 200Hz. The transition bandwidths of the i-th filter are proportional to the i-th filter bandwidth. Thus the last (highest) filter has the widest transition band. The length of the individual filters is specified to be in the range of 139 to 19 samples, with the aim to keep the product 'filter-length transition-bandwidth' constant. This selection of filter lengths assures equal performance for all the five filters in the bank, while having an average complexity of a filter-bank with five filters having each the length of 83 samples. The sampling frequency is specified to be 8000Hz and in the optimization process all the weights were chosen to be equal ($C_i=1$). The power of the input signal to each of the filters was set to be unity, whereas the power of the input noise to the filters was set to be four in the lower stopband and nine in the upper stopband of all filters. Table I summarizes the exact passband/stopband frequencies of the individual filters, their lengths and the SNR in dB, obtained for a Wiener filter bank and for the optimal filter bank designed using (8). The SNR values for the optimal filter bank are seen to be degraded by 10-15dB. However, the composite response of the Wiener filter bank (solid line in Fig. 1), has a ripple of 4dB, compared to the flat composite response of the optimal filter bank, i.e. 0dB ripple (dashed line in Fig. 1).

The frequency response of each of the optimal filters is illustrated in Fig. 2a, and the frequency responses of the Wiener filters in Fig. 2b, both on a linear magnitude scale. The frequency response of the

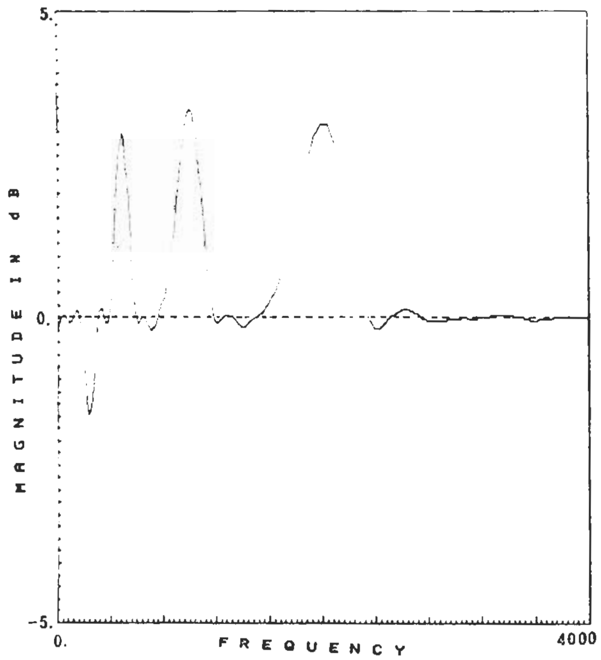

Fig. 1: Composite response of the Wiener filter bank (solid line), and the optimal filter bank (dashed line).

fourth filter in the optimal filter bank is illustrated in Fig. 3a, and for comparison, the frequency response of the fourth Wiener filter is illustrated in Fig. 3b.

The above design example illustrates the strength of the new design method in that it guarantees the specified composite response, even if the individual filters have different length. The composite response can be specified to be flat, as in the above example, or it can be any other desired response (e.g., when the sampling frequency of the input process is higher than the Nyquist rate, a lowpass type of composite response can be specified). Furthermore, the exact specifications that guarantees linear-phase optimal individual filters, were given. For the important special case of uniform filter banks, two types of specifications were considered. When the specifications are in terms of real individual filters, the conventional design approaches which are based on a lowpass prototype filter are necessarily sub-optimal. However, when these specifications are in terms of complex individual filters, the optimal solution, according to the new method, can be expressed in terms of frequency translations of a lowpass filter prototype, which simplifies the design process. It is also shown that this optimal prototype filter has real coefficients and linear phase. The new design method is derived in this work from a statistical point of view, as the maximization of the weighted harmonic mean of the output SNR's of the individual filters, while having a specified composite response. The new design method has also the deterministic interpretation of a WMMSE design of the individual filters in the bank, subject to the composite response specification [2]. In this work the relations between the statistical and the deterministic interpretations of this method were elaborated.

The extension of this method to general complex filter banks is not difficult. A generalization of this method for the design of filter banks with composite response specifications having a given tolerance, is now in progress.

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Table I

Individual Filter's Number	Filters Length	Lower Stopband Frequencies Hz	Passband Frequencies Hz	Higher Stopband Frequencies Hz	$B_i(5)$ [dB]	$\frac{S_{o_i}}{N_{o_i}}(9)$ [dB]
1	139	-	0-200	300-4000	32.94	21.35
2	139	0-200	300-500	600-4000	28.41	18.97
3	79	0-400	600-1000	1200-4000	35.14	20.81
4	39	0-800	1200-2000	2400-4000	34.78	22.08
5	19	0-1600	2400-4000	-	38.57	28.43

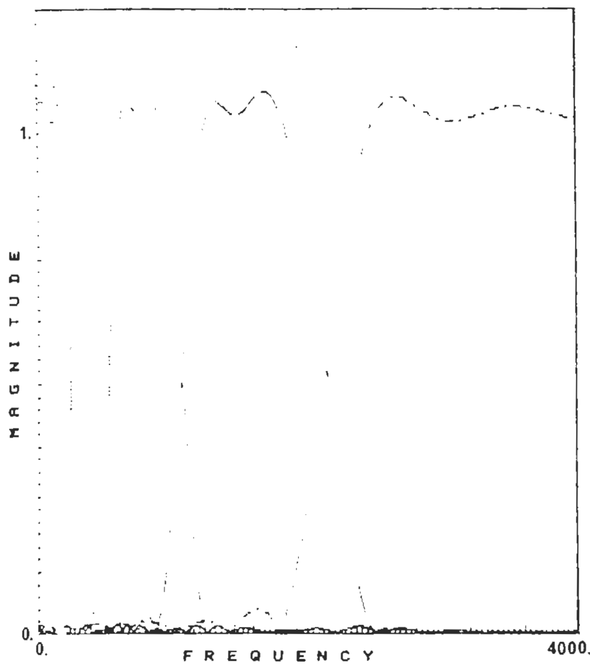

Fig. 2a: Frequency response of each of the five optimal filters (linear magnitude scale).

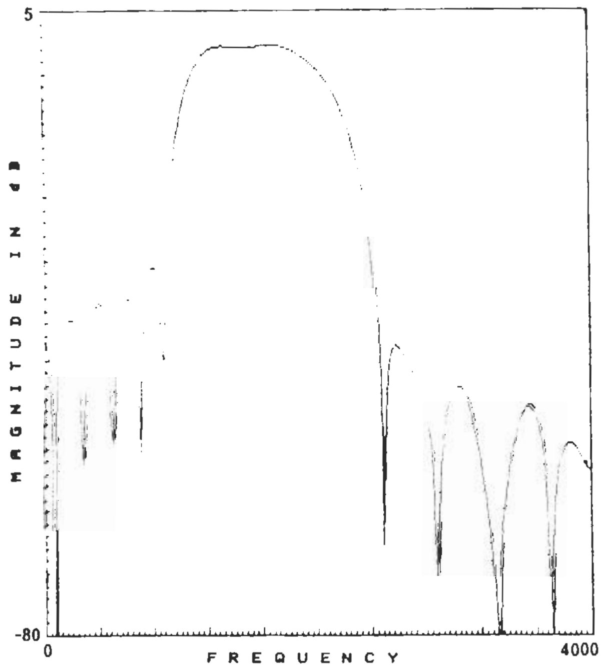

Fig. 3a: Frequency response of the fourth optimal filter (in dB).

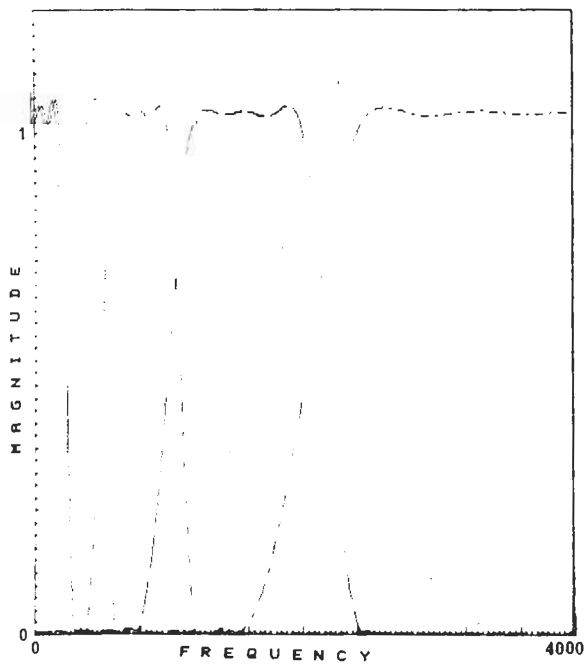

Fig. 2b: Frequency response of each of the five Wiener filters (linear magnitude scale).

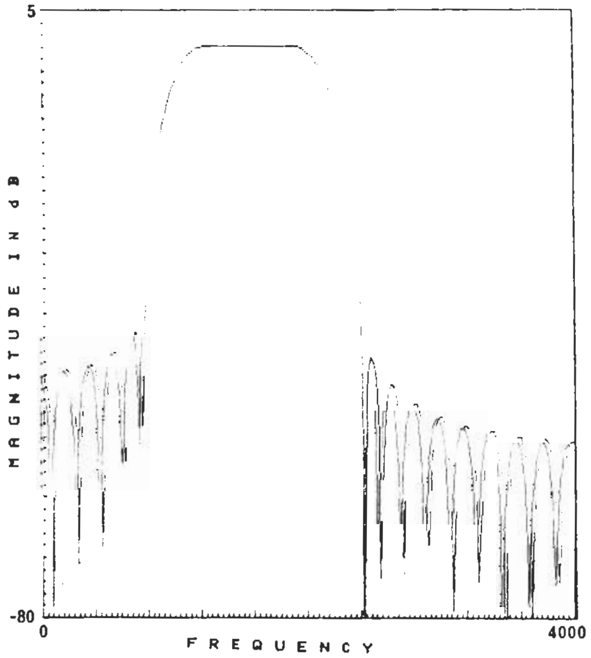

Fig. 3b: Frequency response of the fourth Wiener filter (in dB).

Appendix A

Combining (1), (2) and (7b) the optimization criterion is:

$$\underset{\{\underline{a}_i\}_{i=1}^{N}\in S_c}{Min}\left\{\sum_{i=1}^{N}\frac{C_i}{r_{ss_i}(o)}\left(r_{ss_i}(o)-2\underline{a}_i^T R_{ss_i}\cdot\underline{u}_{d_i}+\underline{a}_i^T R_{xx_i}\underline{a}_i\right)\right\} \quad (A1)$$

with the following definitions:

$$W_i(f)^2 \triangleq F\{r_{xx_i}(k)\} \;\; ; \;\; D_i(f)\triangleq F\{r_{ss_i}(k-d_i)\}/\,W_i(f)^2 \;\; ;$$

$$H_i(f) \triangleq F\{a_i(k)\} \;\; ; \;\; K_i^2 \triangleq \frac{C_i}{r_{ss_i}(o)} \;\; ; \quad (A2)$$

where $F\{\cdot\}$ denotes Fourier transformation, (A1) is equivalent to the following minimization problem:

$$\underset{\{\underline{a}_i\}_{i=1}^{N}\in S_c}{Min}\left\{\sum_{i=1}^{N}K_i^2\int_{-0.5}^{0.5}W_i(f)^2|D_i(f)-H_i(f)|^2df\right\} \quad (A3)$$

This is easily seen to be similar to the WMMSE criterion defined in [2].

The following relations between the statistical and the deterministic functions are the outcome of (A2): The input spectrum is related to the weight function in the WMMSE, the desired output signal spectrum divided by the input spectrum is related to the desired frequency response in the WMMSE, and the relative weight constant C_i, is related to the relative weight constant (K_i^2) in the WMMSE. The solution to (A3) (which is the solution to (A1)), was already given in [2]. We shall briefly summarize the solution as follows:

(a). Under the following change of variables:

$$\underline{a}_i = \underline{a}_i^{*} + \underline{b}_i \qquad , \quad i=1,\ldots,N \quad (A4)$$

(A1) becomes in terms of the new variables:

$$\underset{\{\underline{b}_i\}_{i=1}^{N}}{Min}\left\{\sum_{i=1}^{N}\frac{C_i}{r_{ss_i}(o)}\underline{b}_i^T\cdot R_{xx_i}\cdot\underline{b}_i\right\} \quad (A5)$$

subject to:

$$\sum_{i=1}^{N}\underline{b}_i^{aug} = \underline{a}_c^{*}-\sum_{i=1}^{N}\underline{a}_i^{*aug} \;\; (\triangleq \underline{p}) \quad (A6)$$

where $\underline{a}_i^{*}$, $i=1,\ldots,N$, are the Wiener filters, obtained from (4), and $\underline{a}_c^{*}$ is the desired composite filter.

(b). Solving (A5) and (A6), by introducing a Lagrange multiplier λ, and differentiating with respect to the $\{\underline{b}_i\}_{i=1}^{N}$, results in the unique optimal solution:

$$\frac{C_i}{r_{ss_i}(o)}R_{xx_i}\cdot\underline{b}_i=\lambda\cdot\left(\sum_{i=1}^{N}\underline{b}_i^{aug}-\underline{p}\right)|_{M_i}\;(\triangleq\underline{q}\,|_{M_i}) \quad , \; i=1,\ldots,N \quad (A7)$$

where the value of λ is set so that (A6) holds. By simple algebra, (A7) and (A4) result in (8) (R_{xx_i} is p.d., thus non-singular).

(c). Substitution of the solution of (8), in (2) (which gives the definition of N_{o_i}), and using (1), results immediately in the optimal output SNR value, given in (9).

(d). The value of $\underline{q}$ is obtained by substituting the solution (8) in the constraint (6), and is given by:

$$\underline{q} = \left[\sum_{i=1}^{N}\frac{r_{ss_i}(o)}{C_i}(R_{xx_i}^{-1})^{aug}\right]^{-1}\cdot\underline{p} \quad (A8)$$

Matrix augmentation from $M_i{\times}M_i$ dimension to $M_c{\times}M_c$ dimension, follows from the corresponding vector augmentation definition in (6). The rows are first augmented as vectors in $\mathbb{R}^{M_i}$ to $\mathbb{R}^{M_c}$, and then the columns are augmented in the same manner.

(A8) and (A6) together with (8) and (9) define the solution completely.

If the Wiener filters resulting in (4), satisfy the constraint (6), it follows from (A6) that $\underline{p}=\underline{o}$, and then from (A8) that $\underline{q}=\underline{o}$. Thus, in this case the solution of the new method (in (8) and (9)) coincides with the Wiener filters solution in (4).

We shall now prove that (10) is a sufficient condition for all N optimal individual filters to have a linear phase.

(a). Since the matrices R_{xx_i} and R_{ss_i} defined in (3), are toeplitz symmetric matrices, and thus are also centro-symmetric matrices (i.e. symmetric with respect to both main diagonals), it follows that for d_i which are set according to (10a), $\underline{a}_i^{*}$ has even symmetry.

(b). Since (10) is satisfied, the vectors $\underline{a}_c^{*}$ and $\{\underline{a}_i^{*}\}_{i=1}^{N}$ have even symmetry, and it follows that the vector $\underline{p}$ defined by (A6) has also even symmetry (the augmentation does not harm the symmetry).

(c). Since R_{xx_i} are centro-symmetric matrices, so are $R_{xx_i}^{-1}$. The augmented linear combination of the matrices $R_{xx_i}^{-1}$ in (A8) preserves the centro-symmetric property of its components. The vector $\underline{p}$ is symmetric, thus from (A8), the vector $\underline{q}$ is also symmetric.

(d). Since $\underline{q}$ is symmetric, $\underline{q}\,|_{M_i}$ is also symmetric (providing of course (10a) and (10b) are satisfied). Since $R_{xx_i}^{-1}$ are centro-symmetric, and $\{\underline{q}\,|_{M_i}\}_{i=1}^{N}$ and $\{\underline{a}_i^{*}\}_{i=1}^{N}$ are symmetric, it follows from (8), that all the optimal individual filters $\{\underline{a}_i\}_{i=1}^{N}$ are also symmetric, and therefore have linear phase.

Q.E.D.

Appendix B

We shall prove that the optimal filter bank solution in (8), can be expressed in terms of a frequency translated lowpass prototype, as shown in (14), if the filter bank specifications are according to (13a) - (13d).

(a). From (13d) it follows that:

$$R_{xx_i}^{-1} = \Delta^{-1}\cdot W^{-i}\cdot R_{xx_o}^{-1}\cdot W^{i}\cdot\Delta \quad (B1)$$

and

$$r_{ss_i}(o) = r_{ss_o}(o) \quad , \quad i=1,\cdots,N \quad (B2)$$

(b). Due to (13a) all augmentation and reduction operations are ignored, since all the filters have the same length and signal delay. Substituting (13a)-(13d) along with (B1) and (B2) in the optimal filter bank equations (4), (8), (A6) and (A8), the simplified solution is given by the following equations:

$$\underline{a}_i = \Delta^{-1}\cdot W^{-i}(R_{xx_o}^{-1}\cdot R_{ss_o})W^{i}\cdot\Delta\cdot\underline{u}_{d_c} + \quad (B3)$$

$$+\, r_{ss_o}(o)\cdot\Delta^{-1}\cdot W^{-i}\cdot R_{xx_o}^{-1}\cdot W^{i}\cdot\Delta\cdot\underline{q}$$

$$\underline{q} = \frac{\Delta^{-1}}{r_{ss_o}(o)}\left(\sum_{i=1}^{N}W^{-i}\cdot R_{xx_o}^{-1}\cdot W^{i}\right)^{-1}\cdot\Delta\cdot\underline{p} \quad (B4)$$

$$\underline{p} = \left[I-\Delta^{-1}\cdot\sum_{i=1}^{N}W^{-i}\cdot(R_{xx_o}^{-1}\cdot R_{ss_o})\cdot W^{i}\cdot\Delta\right]\cdot\underline{u}_{d_c} \quad (B5)$$

(B3) follows from (8), (B4) from (A8), and (B5) from (A6).

(c). We now state several properties that will ease our proof:
An important property of the matrix W is the following:

a_c-poly vectors. Furthermore $\underline{u}_{d_c}$ is an eigenvector of Δ, with eigenvalue $\omega_N^{\Delta d_c}$. Applying (B8) to $\Delta\cdot\underline{q}$ and $\Delta\cdot\underline{u}_{d_c}$, (B3) becomes:

$$\underline{a}_i=\omega_N^{(i+\Delta)d_c}\Delta^{-1}\cdot W^{-i}\{R_{xx_o}^{-1}[R_{ss_o}\cdot\underline{u}_{d_c}+r_{ss_o}(o)\cdot\omega_N^{-\Delta d_c}\cdot\Delta\cdot\underline{q}]\} \quad (B9)$$

Letting $\hat{\underline{q}}\triangleq\omega_N^{-\Delta d_c}\cdot\Delta\cdot\underline{q}$, (B9) becomes the union of (14) and (15).

Q.E.D

Since in both (15) and (16) all the terms are real, it is enough to prove (16), in order to prove that $\underline{a}_o$ has real coefficients.

As a first step in the proof of (16), we combine (B4), (B5) and the above definition of $\hat{\underline{q}}$ to obtain:

$$\hat{\underline{q}} = \frac{\omega_N^{-\Delta d_c}}{r_{ss_o}(o)}\cdot \quad (B10)$$

$$\cdot\left[\sum_{i=1}^{N}W^{-i}R_{xx_o}^{-1}W^{i}\right]^{-1}\cdot\left[\sum_{i=1}^{N}W^{-i}\left(\frac{1}{N}I-R_{xx_o}^{-1}\cdot R_{ss_o}\right)\cdot W^{i}\right]\cdot\Delta\cdot\underline{u}_{d_c}$$

$\underline{u}_{d_o}$ is eigenvector of Δ with eigenvalue $\omega_N^{\Delta d_c}$, and the two matrices in brackets are N-poly matrices, according to (B6). We shall denote by $(A)_{NP}$, the N-poly matrix

$$B_i=\frac{r_{ss_o}(o)}{r_{ss_o}(o)-\underline{u}_{d_c}^T\cdot R_{ss_o}\cdot R_{xx_o}^{-1}\cdot R_{ss_o}\cdot\underline{u}_{d_o}} \quad , \quad i=1,\ldots,N \quad (B15)$$

Using the definition of $\hat{\underline{q}}$, and the fact that $\overline{\underline{q}}\,\Delta^{-1}=\overline{(\Delta\cdot\underline{q})}$, we get from (B13) and (B15) (using also the fact that $\hat{\underline{q}}$ is real):

$$\frac{S_{o_i}}{N_{o_i}}=\frac{B_o}{1+B_o\cdot r_{ss_o}(o)\cdot\hat{\underline{q}}^{\,T}\cdot R_{xx_o}^{-1}\cdot\hat{\underline{q}}} \quad , \quad i=1,\ldots,N \quad (B16)$$

(B16) proves that the output SNR is the same for all N individual filters, and is given by (17).

Q.E.D.