New Morphological Skeleton Properties Applicable to its Efficient Coding*

R. Kresch and D. Malah

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel. E-mail: kresch@dekel.technion.ac.il

ABSTRACT — **This work presents new theoretical results concerning the Morphological Skeleton Representation of images. Based on these properties, one can considerably improve existing schemes for *Skeleton-Based Coding*, or design new efficient ones.**

An example of such a scheme is also presented. Computer simulations indicate that, typically, the proposed coding scheme substantially improves the coding rates obtained by the best previous schemes for Skeleton coding, and is more efficient than coding the original binary image by Chain Code, Quadtree and Run-length/Huffman methods. Moreover, the related algorithm is fast, when properly implemented.

1 Historical Background

It is well-known that the Skeleton decomposition of images (binary images, mainly) is suitable for Compression [3]. However, the compression rates achieved until now by lossless-coding of the Skeleton were only comparable to (and sometimes even worse than) other simpler methods (such as Chain Coding, Quadtree Decomposition and Run-length Coding) applied directly to the original image. This made many researchers skeptical about Skeleton-based Coding.

On the other hand, little was proposed concerning the improvement of the *coding scheme* itself! [3, 4]

In this work, we present a number of theorems concerning properties of the Skeleton Representation. These properties are not used by conventional Skeleton-coders, and this is reflected in their unsatisfactory performance.

2 Basic Definitions and Notation

The authors suppose the readers to be familiar with the basic translation invariant operations of Mathematical Morphology: dilation ($\oplus$), erosion ($\ominus$), opening ($\circ$), and closing ($\bullet$). For background material see [1, 3].

2.1 Discrete Morphological Skeleton Representation

The theorems in this paper are related to a *discrete* Generalized-Step Skeleton Representation [5] of a given image X. By "discrete" we mean that the family of elements used in the Skeleton decomposition is indexed by *natural* numbers $(0, 1, \ldots)$. On the other hand, X and the shapes in the above decomposition family are not restricted to be discrete. They can be discrete (sets in $\mathcal{Z}^2$), continuous (sets in $\mathcal{R}^2$) or grayscale images (functions over $\mathcal{Z}^2$ or $\mathcal{R}^2$).

A discrete Generalized-Step Skeleton Representation is a collection of subsets $\{S_n\}$, $n = 0, 1, \ldots$, obtained by the following discrete version of Lantuéjoul's formula [5, 6]:

$$S_n = X \ominus A(n) - [X \ominus A(n)] \circ B(n), \qquad (1)$$

where the series $\{B(n)\}$ $(n = 0, 1, \ldots)$ is *any* predefined family of shapes, and the series $\{A(n)\}$ $(n = 0, 1, \ldots)$ is generated from $\{B(n)\}$ in the following way:

$$\begin{cases} A(n+1) = A(n) \oplus B(n), & n = 0, 1, \ldots \\ A(0) = \{(0,0)\}. \end{cases} \qquad (2)$$

As a particular case, $A(n) = nB$ (where B is a fixed structuring element) when we set $B(n) = B$, $\forall n$.

From this point on, we adopt the following notation throughout the paper:

$$X_n \triangleq X \ominus A(n), \qquad (3)$$

$$Y_{n+1} \triangleq X_{n+1} \oplus B(n) = X_n \circ B(n). \qquad (4)$$

This way we can rewrite (1) as $S_n = X_n - Y_{n+1}$.

*This work was supported by the Fund for the Promotion of Research at the Technion.

The Skeleton Subsets $\{S_n\}$ are disjoint, and their union $S \triangleq \bigcup_n S_n$ is called the Skeleton. Each Skeleton point belongs to one and only one Skeleton Subset S_n and its index n is called its *quench value*. The map relating each Skeleton point to its quench value is called *Quench Function* [1].

The original shape X can be *fully* or *partially* reconstructed from the collection of Skeleton Subsets $\{S_n\}$ by the formula [5]:

$$X \circ kB = \bigcup_{n \geq k} S_n \oplus A(n). \tag{5}$$

Full (partial) reconstruction is obtained for $k = 0$ ($k > 0$) in (5).

2.2 Connectivity, Reconstruction and Ultimate Erosions

Connectivity, Reconstruction and Ultimate Erosions are fundamental concepts in this work. Reconstruction and Connectivity are defined in [1] in the following way:

Definition 1 (Connectivity)
Two points x and y are connected (under a predefined structuring element B) iff $\{x\} \subseteq \{y\} \oplus B$ or $\{y\} \subseteq \{x\} \oplus B$.

Definition 2 (Reconstruction) *Let A, D be two sets, such that $D \subseteq A$, and B a pre-defined structuring element. The* Reconstruction *of A from D using B, $Rec_{[B]}\{A, D\}$ is given by the following recursive formula:*

$$Rec_{[B]}\{A, D\} \triangleq \{[(D \oplus B) \cap A] \oplus B\} \cap A \ldots \tag{6}$$

Intuitively, when B is a *symmetric* structuring element, the Reconstruction of A from D is the collection of *connected components* of A which contain points of D. It is important not to confuse the above operation of Reconstruction with the previously mentioned *reconstruction of the original image X from its Skeleton Representation.*

The definition of *Ultimate Erosions* presented bellow in slightly generalized in order to meet our purposes:

Definition 3 (Ultimate Erosions) *The* Ultimate Erosion *of order n, denoted as U_n, of a given image X is the set given by:*

$$U_n \triangleq X_n - Rec_{[C(n)]}\{X_n, Y_{n+1}\} \tag{7}$$

where $C(n) = B(n-1)$, for $n \geq 1$, and $C(0)$ is an arbitrary structuring element.

The original definition in [1] is restricted to the case where $B(n) = B$ (and therefore $A(n) = nB$), hence $C(n) = B$, for all n. Intuitively, the Ultimate Erosions are those Skeleton points with maximal quench value within each region of the original shape. (See the example in Fig. 1(a)). They are usually a small percentage of the Skeleton.

3 New Skeleton Properties

Our main theoretical results are presented in this section. They are related *only* to a **discrete** Generalized-Step Skeleton Representation (see section 2.1).

3.1 Quench Function Sampling

Theorem 1 *The quench values of the Skeleton points corresponding to the* Ultimate Erosions, *in addition to the position of all the Skeleton points, are sufficient for perfect reconstruction.*

Proof (Outline) *We use induction in the following way: (i) If N is the maximal quench value in the Skeleton, then $X_N = U_N$. (ii) Once X_{n+1} is known, each set X_n, $N > n \geq 0$, can be calculated (see bellow), and (iii) the original image X is equal to X_0.*

In order to obtain the second part of the above induction, suppose that X_{n+1} is available. Therefore Y_{n+1} is also available. From the hypothesis, the Skeleton S and the Ultimate Erosions $\{U_n\}$ are provided. Then X_n is obtained from the above (not proven here due to lack of space) by:

$$X_n = U_n \cup Rec_{[C(n)]}\{[Y_{n+1} \cup S], Y_{n+1}\} \tag{8}$$

where $\{C(n)\}$ are as defined in Definition 3. □

The above proof is constructive; it provides a reconstruction algorithm for the original image from the resulting "Quench-sampled" Skeleton. It consists of calculating at each step n, which varies from N down to 0, the set X_n according to (8).

Corollaries 1 and 2 bellow are direct consequences of (8). In both of them, suppose that, for each n, the structuring element $C(n)$ is *symmetric*. In addition, Connectivity is considered under the same $C(n)$.

Corollary 1 *If s is a Skeleton point with quench value n, then all the Skeleton points in the connected component to which it belongs have also quench value n.*

According to Corollary 1, not all the ultimate-erosion points need to have their quench value stored! For every connected component in the set of ultimate erosions, one needs to store only the quench value of *one point*. Note that the set of ultimate erosions is usually a very small subset of the Skeleton points, and, due to the above consideration, only a small percentage of them need to have their quench values stored. This provides a sampling scheme of the Quench Function. (See Fig. 1(b)).

Corollary 2 *A Skeleton point s has quench value n* if and only if *$s \notin Y_{n+1}$ and s is connected to either U_n or Y_{n+1}, directly or by a path of Skeleton points.*

The above corollaries are used in the Coding scheme proposed in section 4.

Corollaries similar to Corollaries 1 and 2 above can be derived also for *asymmetric* structuring elements, if we substitute the concept of *Connectivity* by *Descendance* (see [2]).

3.2 Deterministic Prediction

Suppose a Coding procedure where, at a certain step, the Skeleton Subset of order n, S_n, is to be coded, and that Y_{n+1} is known to be available both to the coder and the decoder. Since $S_n = X_n - Y_{n+1}$, it follows that there are no points of S_n inside the region Y_{n+1}. Therefore the coder does not need to code the *status* (whether belonging, or not, to S_n) of the pixels inside Y_{n+1}, and the decoder does not need to "look for" Skeleton points in that region at that moment. This was used in the coding schemes proposed in [3].

It turns out that there is also a region *outside* Y_{n+1} that can be predicted not to contain Skeleton points from S_n. This region can be characterized by the following theorem:

(a) (b)

Figure 1: (a) Skeleton and Ultimate Erosions of a portion of the image "Coffee Grains". The Ultimate Erosions are the black Skeleton points. (b) A subset of the Ultimate Erosions (the four black points). Their quench values, in addition to the position of all the Skeleton points, are sufficient for perfect reconstruction.

Theorem 2 *Let p be any point in the Euclidean space in which X is defined ($\mathcal{R}^2$ or $\mathcal{Z}^2$). If the following holds:*

$$[(Y_{n+1} \cup \{p\}) \bullet A(n)] \circ B(n) \supset \{p\} \qquad (9)$$

then p cannot belong to S_n.

Proof *The proof is by contradiction. Suppose p is in S_n, and let us define the operator $\rho(Z) \triangleq [Z \ominus A(n)] \circ B(n)$. By definition of Y_{n+1}, $\rho(X) = Y_{n+1}$. Also the set $Y_{n+1} \oplus A(n)$ gives Y_{n+1} when operated upon by $\rho(\cdot)$. Therefore, any set Z_0, such that $Y_{n+1} \oplus A(n) \subseteq Z_0 \subseteq X$, satisfies $\rho(Z_0) = Y_{n+1}$. In particular, $Z_0 = (Y_{n+1} \cup \{p\}) \oplus A(n)$, $p \in S_n$, satisfies it. However, according to (9) $\rho(Z_0) \supset \{p\}$, and, therefore, $p \in Y_{n+1}$, which contradicts that $p \in S_n$.* □

Theorem 2 provides a test for each point in the Euclidean space: If it passes it, i.e. (9) holds, then its status as a Skeleton point need not to be coded because it is known to both coder and decoder to be negative. On the other hand, if the test fails ((9) does not hold), nothing can be said about the point status, and it must be coded.

The above test is however practically inviable, because it is extremely computation-demanding. Luckily, a simplified, much faster test is possible in many cases by the following corollary:

Corollary 3 *Let p be a point in the Euclidean space, and let F be any set* not *containing the origin (denoted as O), and satisfying:*

$$[(F \cup O) \bullet A(n)] \circ B(n) \supset \{O\} \qquad (10)$$

If $(F \oplus \{p\}) \subseteq Y_{n+1}$, then p cannot belong to S_n.

In other words, one can pre-select a template F, excluding the origin and usually containing *few points*, such that it satisfies (10). Since it is independent of the input image X, the above selection is done "off-line", and only once for a given decomposition family $\{B(n)\}$. During an "on-line" Coding algorithm, the "prediction test" is performed, for each point p, by placing the template F "on" p and examining the status of the indicated points.

The points found in the above test are only a *subset* of the "predictable points" found in the test of Theorem 9. In order to find *all* the predictable points, a family $\{F_i\}$ of *all* the templates satisfying (10) should be defined, and the test in Corollary 3 must be repeated for each F_i. This could also be very computation-demanding. Often, however, a small subset of $\{F_i\}$ is enough for

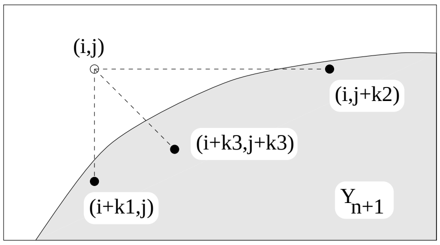

Figure 2: A point (i, j) predicted not to belong to S_n according to Corollary 4.

finding *most* of the desired points. As an example, consider a Skeleton decomposition of a *discrete* set X, having $B(n) = B$, $\forall n$, where B is a 3×3-*pixel squared structuring element.* In this case, Corollary 3 above can assume the following specific format:

Corollary 4 *Let* $(i, j) \in \mathcal{Z}^2$*, and consider a Skeleton with a* 3×3*-pixel* squared *structuring element. If any of the triplets*

$$\begin{array}{l}\{(i+k_1, j), (i, j+k_2), (i+k_3, j+k_3)\},\\ \{(i-k_1, j), (i, j-k_2), (i-k_3, j-k_3)\},\\ \{(i+k_1, j), (i, j-k_2), (i+k_3, j-k_3)\},\\ \{(i-k_1, j), (i, j+k_2), (i-k_3, j+k_3)\},\end{array}$$

for any integers k_1*,* k_2 *and* k_3 *in the interval* $[2, 2n+1]$*, is contained in* Y_{n+1}*, then the point* (i, j) *does not belong to* S_n*.*

The above triplets represent a subset of the family $\{F_i\}$ related to the given squared structuring element. Fig. 2 shows an example of a point (i, j) which is predicted not to belong to S_n in this specific case.

4 Proposed Coding Scheme

In this section we propose an efficient coding scheme of the Skeleton Representation, which makes use of the properties presented above. In general lines, the proposed algorithm is as follows. After the Skeleton Representation is calculated, the coding is performed in the same way as the decoding, i.e., by reconstructing the original image. Let N be the maximum quench value. Initially, for each of the Ultimate Erosions U_n, $0 \le n \le N$, a set $\tilde{U}_n$ is formed, containing one point of each connected component of U_n. Then, the points in the above sets have their position and quench value coded. At this point, the main loop starts. At each step n, which varies from its maximum value, N, down to 0, a scanning procedure is performed on the *external boundary* of Y_{n+1} and of $\tilde{U}_n$ (the external boundary is provided here by the operator $(\cdot) \oplus C(n) - (\cdot)$, $\forall n$). Only these external boundaries need to be scanned, since the Skeleton points in S_n are necessarily linked either to Y_{n+1} or to $\tilde{U}_n$ (Corollary 2). Some points in the above scan can be predicted not to belong to S_n by the test in Corollary 3. These points are skipped. The other (non-predictable) points have their status ("true" or "false") as a Skeleton point coded by an arithmetic coder. If a point has status "true", its neighborhood is searched for other connected Skeleton points in a recursive way, before the main scanning procedure goes on. This procedure is detailed in [7].

The above algorithm is shown, in simulations on *binary images*, to outperform one of the best previous Skeleton coders, presented in [4], and the well-known Ziv-Lempel, Run-length + Huffman, Quadtree and Chain coders, and its complexity is on par with their complexity [7]. At this point, it is still weaker than the most advanced Standards for facsimile (G4 and JBIG), but it is comparable to the 2-dimensional Group 3 Standard (G3D2, with $k = 4$), being usually more efficient than it.

5 Conclusion

We have presented new Skeleton properties, and shown how to efficiently take advantage of them in order to design a fast and efficient coder of binary images via their Skeleton Representation.

The presented theoretical results can be applied also to grayscale images and "labeled" images (obtained by a segmentation procedure).

References

[1] J. Serra, *Image Analysis and Mathematical Morphology,* London: Academic Press, 1982.

[2] J. Serra, *Image Analysis and Mathematical Morphology, Vol.2: Theoretical Advances.* New York: Academic Press, 1988.

[3] P. Maragos and R.W. Schafer, "Morphological Skeleton Representation and Coding of Binary Images", *IEEE Trans. ASSP*, Vol. 34, No. 5, pp.1228-1244, October 1986.

[4] J.W. Brandt, A.K. Jain and V.R. Algazi, "Medial Axis Representation and Encoding of Scanned Documents", *JVCIR*, Vol. 2, No. 2, pp.151-165, June 1991.

[5] D. Schonfeld and J. Goutsias, "Morphological Representation of Discrete and Binary Images", *Trans. Signal Processing*, Vol. 39, No. 6, pp.1369-1379, June 1991.

[6] R. Kresch and D. Malah, "Morphological Reduction of Skeleton Redundancy", *Signal Processing*, Vol. 38, pp. 143-151, August 1994.

[7] R. Kresch and D. Malah, "An Efficient Coding Scheme for Binary Images Based on the Morphological Skeleton Representation", *Proc. IEEE 18th Convention of Electrical and Electronics Engineers in Israel*, Tel-Aviv, March 1995.