USING IMPLICIT POLYNOMIALS FOR IMAGE COMPRESSION

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Abstract

Implicit polynomials (IP) are being used to represent 2D curves and 3D surfaces [1,2,3,5]. The zero-set of a 2D implicit polynomial of the form of $p(x,y)=a_1x^N+a_2x^{N-1}y+...+a_r=0$ can be used to describe data points making up a 2D curve. A similar 3D polynomial - $p(x,y,z)=0$ describes data points on a 3D surface [2]. In this paper we describe a way to restrict the zero-set of the fitted polynomial so that correct restoration of the data from the polynomial's coefficients will be achieved.

1. Introduction

The use of IPs in image processing is common in object recognition [4]. Because IPs can describe complicated shapes with few coefficients, they are attractive for image compression as well. In image compression applications [6], IPs can be used for the representation and coding of contour information for shape-adaptive image coding.

One main obstacle in the usage of IPs for image compression is the difficulty in forcing the zero set of an IP to be limited only to where there is data. Leading fitting algorithms produce polynomials whose zero-set attempts to contain the data points, but there is no guaranty that zero-set points would not exist where there is no data.

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The value of the polynomial at a point (x,y) can be described as the product of two vectors – a parameter vector (containing the polynomial's coefficients), and a vector of monomials.

For a d[th] degree polynomial, the monomial vector is denoted as:

$$\overline{p}(x,y) = \left[x^0 y^0, x^1 y^0, x^0 y^1, \ldots, x^d y^0, x^{d-1} y^1, \ldots, x^1 y^{d-1}, x^0 y^d\right] \tag{1}$$

and the parameter vector is

$$\overline{a} = [a_1, a_2, \ldots, a_r] \tag{2}$$

where $r = (d+1)(d+2)/2$.

The value of the polynomial described by $\overline{a}$ at location (x,y) is:

$$P_{\overline{a}}(x,y) = \overline{a}\,\overline{p}^T(x,y) \tag{3}$$

In order to describe the boundary data with the polynomial, we are looking for a parameter vector $\overline{a}$ that leads to a polynomial that best fits the data and allows data restoring with minimal error.

For image compression applications, where data rate and distortion are of interest, the fitting problem is that of finding the coefficient vector which minimizes the distance between data points and the zero-set of the polynomial, with a resulting polynomial that is as robust as possible to coefficient quantization. The fitting problem can be described as a minimization problem of finding an optimal coefficients vector $\overline{a}_{OPT}$, which minimizes a squared error E defined by [7]:

$$E = \overline{e}\,\overline{e}^T \tag{4}$$

where

$$\overline{e} = (\overline{a}M - \overline{b}) \tag{5}$$

with the following definitions:

$$\begin{aligned} \overline{b} &= \left[\overline{0} \quad \overline{dx} \quad \overline{dy}\right] \\ M &= \left[M_0 \quad M_X \quad M_Y\right] \end{aligned} \tag{6}$$

where $\overline{dx}, \overline{dy}$ are vectors defining the required differentials (see below) of the polynomial at the data points and,

$$\begin{aligned} M_0 &= \left[\overline{p}^T(x_1,y_1) \ \ldots \ \overline{p}^T(x_N,y_N)\right] \\ M_X &= \left[\overline{p}_X{}^T(x_1,y_1) \ \ldots \ \overline{p}_X{}^T(x_N,y_N)\right] \\ M_Y &= \left[\overline{p}_Y{}^T(x_1,y_1) \ \ldots \ \overline{p}_Y{}^T(x_N,y_N)\right] \end{aligned} \tag{7}$$

with,

$$\bar{p}_X(x_n, y_n) = \frac{d}{dx}\bar{p}(x_n, y_n)\ ;\ \bar{p}_Y(x_n, y_n) = \frac{d}{dy}\bar{p}(x_n, y_n) \tag{8}$$

According to [7], the vectors $\overline{dx}$ and $\overline{dy}$ in (6) characterize the fitting algorithm. To obtain a fitting algorithm which minimizes the max error (denoted as *Min-Max* fitting algorithm), $\overline{dx} = [dx_1, ..., dx_n]$ and $\overline{dy} = [dy_1, ..., dy_n]$ should satisfy:

$$\begin{aligned} &\frac{dy_n}{dx_n} = tg(\alpha_n) \\ &\sqrt{\left(dx_n^{\ 2} + dy_n^{\ 2}\right)} = \sum_{k=1}^{r} \left|p_k(x_n, y_n)\right| \end{aligned} \tag{9}$$

where α_n is the angle between the local perpendicular to the data-set around point (x_n, y_n) and the x axis.

A Least Squares (LS) solution of the above fitting problem is:

$$\bar{a}_{OPT} = \bar{b}M^T\left(MM^T\right)^{-1} \tag{10}$$

However, solving the LS problem with the parameters $\bar{b}$ and M leads to a polynomial with optimal coverage of the data points, but with no consideration for spurious zero-set points that don't describe any given data points. Therefore, when reconstructing the compressed data from the coefficients describing an IP, zero-set points that exist where there is no data lead to reconstruction errors.

2. Constrained fitting

Previous work addressing the this problem [5] attempts to limit the possible space of polynomials to special groups or star shaped polynomials, which prevent spurious zero-set points. Our approach is to prevent the occurrence of spurious zero-set points near the original data and use the continuity property of the zero-set in order to restore the data, therefore, neglecting far spurious zero-set points.

In order to recover the data set from the parameters (polynomial coefficients) it should be decided which of the zero-set points represent the original data.

Implementation of the following requirements allow a unique restoration of the data:

- The zero-set of the polynomial has to be continuous along adjacent data points.
- There should be no splits or intersections of the zero-sets of the polynomial.
- A starting point is supplied for restoration.

This set of requirements assures us that if we begin scanning for zeros at the given starting point and move continuously along the zero-set, we can restore the data and stop when we have reached the starting point again.

The two first conditions listed above can also be stated as restrictions on the values of the polynomial.

We demand that the zero-set of the polynomial lie within a thin strip surrounding the data set (see Fig. 1). This means that the value of the polynomial around the strip cannot be zero, i.e. it must be positive or negative for external or internal points respectively as depicted in Fig. 1.

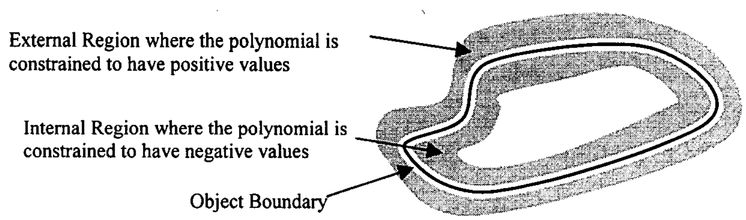

Fig. 1. – Original data set and constraint regions

These restrictions, dependent on the thickness of the constraint strips, in addition to the fact that the zero-set of the polynomial is continuous, can prevent discontinuities and crossings or intersections of the zero-set.

To implement these requirements we use a *constrained least squares* solution of (10) with the constraints specified in (11). The fitting problem becomes:

Find $\bar{a}$ which minimizes $E = \bar{e}\bar{e}^T$ where $\bar{e} = (\bar{a}M - \bar{b})$ subject to:

$$\begin{aligned} &\bar{a}M_{EXT} > \bar{0} \\ &\bar{a}M_{INT} < \bar{0} \\ &M_{EXT} = \left[\bar{p}^T\left(x_{EXT_1}, y_{EXT_1}\right) \;\; \ldots \;\; \bar{p}^T\left(x_{EXT_{N-EXT}}, y_{EXT_{N-EXT}}\right)\right] \\ &M_{INT} = \left[\bar{p}^T\left(x_{INT_1}, y_{INT_1}\right) \;\; \ldots \;\; \bar{p}^T\left(x_{INT_{N-INT}}, y_{INT_{N-INT}}\right)\right] \end{aligned} \tag{11}$$

The constraint points $[\left(x_{EXT_1}, y_{EXT_1}\right) \ldots \left(x_{EXT_{N-EXT}}, y_{EXT_{N-EXT}}\right)]$ are points sampled in the external region and $[\left(x_{INT_1}, y_{INT_1}\right) \ldots \left(x_{INT_{N-INT}}, y_{INT_{N-INT}}\right)]$ are points sampled in the internal region (see Fig. 1).

We used *Lagrange multipliers* to solve the constrained least squares problem.

3. Simulation results

Using the constrained least squares solution we obtain a polynomial whose zero set can be used for representing object boundaries for image compression applications. Fig. 2 compares the resulting polynomial fit and consequent restored data of the same boundary using both unconstrained and constrained fits.

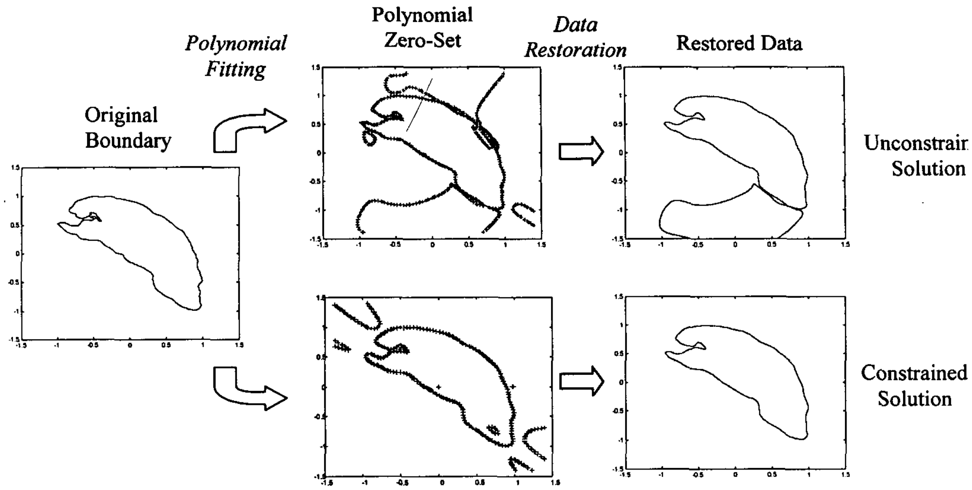

Fig. 2. – Polynomial fitting and data restoration
left – original boundary, center – 14th order fitted polynomial, right – restored data
top – unconstrained *Min-Max* fitting algorithm, bottom – *constrained Min-Max* fitting algorithm

4. Conclusions

In summary, we described the problem of spurious zero-set points that is inherent to unconstrained fitting of data points by implicit polynomials, and provided a constraining scheme for controlling the possible location of these points so that they would not cause data reconstruction errors.

5. References

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