For the security of $\mathcal{K}$, note that $\mathcal{X}$ is assumed to have min-entropy b even given $\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{A}_u, \mathcal{B}_u$, and we can assume that $\mathcal{X}, \mathcal{X}_{i_1}, \mathcal{X}_{i_2}, \ldots, \mathcal{X}_{i_{t-1}}$ are mutually independent as they are generated in distant places. Thus, Theorem 2 implies

$$\|\langle \mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_{t-1}}, \mathcal{K}\rangle - \langle \mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_{t-1}}, \mathcal{U}\rangle\| \leq 2^{-(b+k+2-n-m)/2}$$

where $k = \max(\mathrm{H}_\infty(\mathcal{X}_{i_1}), \ldots, \mathrm{H}_\infty(\mathcal{X}_{i_{t-1}}))$. That is, $\mathcal{K}$ is secure enough when $b + k \gg n + m$. Note that any strong extractor will also work.

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REFERENCES

[1] B. Barak, R. Impagliazzo, and A. Wigdersom, "Extracting randomness from few independent sources," in *Proc. IEEE 45th Annu. IEEE Symp. Foundations of Computer Science (FOCS'04)*, Rome, Italy, Oct. 2004, pp. 384–393.

[2] B. Chor and O. Goldreich, "Unbiased bits from sources of weak randomness and probabilistic communication complexity," *SIAM J. Comput.*, vol. 17, no. 2, pp. 230–261, Apr. 1988.

[3] Y. Dodis, A. Elbaz, R. Oliveira, and R. Raz, "Improved randomness extraction from two independent sources," in *Proc. APPROX-RANDOM*, Cambridge, MA, Aug. 2004, pp. 334–344.

[4] Y. Dodis and R. Oliveira, "On extracting private randomness over a public channel," in *Proc. APPROX-RANDOM*, Princeton, NJ, Aug. 2003, pp. 252–263.

[5] R. Graham and J. Spencer, "A constructive solution to a tournament problem," *Canad. Math. Bull.*, vol. 14, pp. 45–48, Jan. 1971.

[6] R. Impagliazzo, L. A. Levin, and M. Luby, "Pseudo-random generation from one-way functions," in *Proc. 21st Annu. ACM Symp. Theory of Computing (STOC'89)*, Seattle, WA, May 1997, pp. 290–294.

[7] J. Kamp and D. Zuckerman, "Deterministic extractors for bit-fixing sources and exposure-resilient cryptography," in *Proc. IEEE 44th Annu. IEEE Symp. Foundations of Computer Science (FOCS'03)*, Cambridge, MA, Oct. 2003, pp. 92–101.

[8] R. Konig and U. Maurer, "Extracting randomness from generalized symbol-fixing and Markov sources," in *Proc. IEEE Int. Symp. Information Theory (ISIT'04)*, Chicago, IL, Jun./Jul. 2004, p. 232.

[9] C. Lu, O. Reingold, S. Vadhan, and A. Wigderson, "Extractors: Optimal up to constant factors," in *Proc. 35st Annu. ACM Symp. Theory of Computing (STOC'03)*, San Diego, CA, Jun. 2003, pp. 602–611.

[10] N. Nisan and D. Zuckerman, "Randomness is linear in space," *J. Comp. Syst. Sci.*, vol. 52, no. 1, pp. 43–52, Feb. 1996.

[11] R. Shaltiel, "Recent developments in explicit constructions of extractors," in *Bull. EATCS*, vol. 77, 2002, pp. 67–95.

Bounds on the Performance of Vector-Quantizers Under Channel Errors

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***Abstract*—Vector quantization (VQ) is an effective and widely known method for low-bit-rate communication of speech and image signals. A common assumption in the design of VQ-based communication systems is that the compressed digital information is transmitted through a perfect channel. Under this assumption, quantization distortion is the only factor in output signal fidelity. Moreover, the assignment of channel symbols to the VQ reconstruction vectors is of no importance. However, under physical channels, errors may be present, causing degradation in overall system performance. In such a case, the effect of channel errors on the coding system performance depends on the index assignment of the reconstruction vectors. The index assignment problem is a special case of the Quadratic Assignment Problem (QAP) and is known to be NP-complete. For a VQ with N reconstruction vectors there are $N!$ possible assignments, meaning that an exhaustive search over all possible assignments is practically impossible. To help the VQ designer, we present in this correspondence lower and upper bounds on the performance of VQ systems under channel errors, over all possible assignments. The bounds coincide with a general bound for the QAP. Nevertheless, the proposed derivation allows us to compare the bounds with published results on VQ index assignment. A related expression for the average performance is also given and discussed. Special cases and numerical examples are given in which the bounds and average performance are compared with index assignments obtained by known algorithms.**

***Index Terms*—Channel coding, index assignment (IA), performance bounds, vector quantization (VQ).**

I. INTRODUCTION

Vector quantization (VQ) is a method for mapping signals into digital sequences. A typical VQ-based communication system is shown in Fig. 1.

A discrete-time *source* emits signal samples over an infinite (or densely finite) alphabet. These samples should be sent to the *destination* with the highest possible fidelity. The *VQ encoder* translates source output vectors into *channel* digital sequences. The *VQ decoder*'s goal is to reconstruct source samples from this digital information. Since the analog information cannot be perfectly represented by the digital information some *quantization distortion* must be tolerated.

In each channel transmission, the VQ encodes a K-dimensional vector of source samples $\underline{x}(t)$ into a *reconstruction vector index* $y(t)$, where the discrete variable t represents the time instant or a channel-use counter. The index is taken from a finite alphabet $y(t) \in \{0, 1, \ldots, N-1\}$, where N is the number of reconstruction vectors (hence the number of possible channel symbols).

The *index assignment* (IA) is represented in Fig. 1 by a permutation operator

$$\Pi : y(t) \in \{0, 1, \ldots, N-1\} \rightarrow z(t) \in \{0, 1, \ldots, N-1\}. \quad (1)$$

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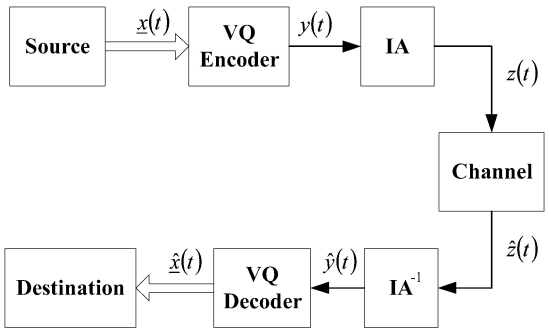

Fig. 1. VQ-based communication system.

The number of possible permutations $N!$ increases very fast with N. For example, for a VQ with just 4-bits index representation ($N = 16$ indices) there are $16! \approx 2 \cdot 10^{13}$ possible permutations. For typical values of N, examination of all possible permutations is therefore impractical. The channel index $z(t) = \Pi\{y(t)\}$ is sent through the channel.

For *memoryless channels*, the channel output $\hat{z}(t)$ is a random mapping of its input $z(t)$, characterized by the *channel probability matrix* H, defined by

$$\{H\}_{ij} = \text{Prob}\{\hat{z}(t) = j \mid z(t) = i\}. \tag{2}$$

Throughout, we assume that H is symmetric (i.e., we consider memoryless channels with a symmetric transition matrix).

For the special case of the *binary-symmetric channel* (BSC)

$$\begin{aligned}\{H\}_{ij} &= \text{Prob}\{\hat{z}(t) = j \mid z(t) = i\} \\ &= q^{d_H(i,j)}(1-q)^{L-d_H(i,j)}\end{aligned} \tag{3}$$

where L is the number of bits ($N = 2^L$) per channel use, q is the *bit-error rate* (BER), and $d_H(i,j)$ is the *Hamming distance* between the binary representations of i and j.

At the receiver, after inverse-permutation, the *VQ decoder* converts the channel output symbols into one of N possible reconstruction vectors. The decoder's output $\underline{\hat{x}}(t)$ is, hopefully, "close" to the original input. The term "close" will be defined by a distortion measure $d(\underline{x}, \underline{\hat{x}})$ between the input and the output of the VQ system.

Knowledge of the source statistics $p(\underline{x})$ or the availability of a representing *training sequence* is assumed. The performance of the overall system is measured in terms of the average distortion $E[d(\underline{x}, \underline{\hat{x}})]$. When a training sequence is used, the average distortion is estimated.

In "classic" discussions of VQ applications, the channel is assumed to be noiseless ($H = I$, where I is the unity matrix) [1], so that no errors occur during transmission and $y(t) = \hat{y}(t)$ for every t. The permutation Π has no effect in this case. This assumption is based upon using a channel encoder–decoder pair to correct channel errors, causing the distortion due to channel errors to be negligible.

Upon knowledge of the source statistics, Lloyd's algorithm [1] may be used to design the VQ. In practice, a training sequence is used and the Linde–Buzo–Gray (LBG) algorithm [1] is implemented. Both methods are iterative and alternately apply the *nearest neighbor* condition and the *centroid* condition.

In some applications, channel coding is not utilized due to its complexity or because of bit-rate constraints. In case of a channel error event, a wrong reconstruction vector is selected at the decoder. The distortion due to channel errors can be significant and affects the design of the VQ system [2]–[17].

In the literature, two main approaches are proposed to improve the performance of vector quantizers under channel errors. The first method allows modification of the partition regions and their corresponding codevectors. In the presence of channel errors, and given the transmitted symbol, the received symbol is a random variable. It is suggested to redesign the VQ by modifying the distortion measure to take all possible output vectors into consideration. This modification results in a *weighted-nearest-neighbor* and *weighted-centroid* conditions [8], [9], [11], [30]. These conditions are specific to every channel condition. Hence, a VQ designed for a noisy channel should, in principle, monitor channel conditions, and apply a different partition and a different set of codevectors for each possible BER. Other drawbacks of this approach are the large memory consumption and extensive design effort.

The second approach tries to reduce channel distortion by using better IAs. The search for the optimal IA is a special case of the Quadratic Assignment Problem (QAP) and is known to be NP-complete [10].

Several suboptimal methods are suggested in the literature. In [12], [13], an iterative algorithm is proposed. After selecting an initial assignment, the algorithm searches for a better assignment by exchanging indices of codevectors, and keeping the new assignment if it performs better than its predecessor. This algorithm can only offer a local minimum. A more sophisticated algorithm is examined in [9], where simulated annealing (SA) is used to search for an optimal IA. The method of SA involves some *ad hoc* arguments to define system "temperature" and "cooling" procedures. Moreover, the method of SA has a very slow convergence rate, and cannot assure global optimum during a limited design period. A suboptimal quadratic placement algorithm [19] is used in [20] for obtaining an efficient VQ IA. Implementation of a search approach for quadratic assignment problems, known as Tabu, is examined in [21]. The Tabu search begins with a random assignment. In each iteration it finds a new solution by making a local step (e.g., index switching). Normally, such local steps will locate a local minima and cycle about it. To avoid cycling, the method records its moves in one or more Tabu lists. This insures that new regions of the problem solution space will be investigated. Similar to the SA, the Tabu search has a slow convergence rate and cannot assure a global optimum.

For the special case of a uniform scalar quantizer and a uniform source under the BSC, it is shown in [28] that the natural binary code (NBC) assignment is an optimal assignment. Later approaches, using eigenvalue arguments, [3], [6], [7] have reached the same conclusion. The NBC is also optimal for the 4-bit uniform scalar quantizer using a $(4, 7)$ Hamming error control code, under the BSC channel [33], [3].

The difficulty in obtaining good assignments validates our development of performance bounds. These bounds and a related expression for the average performance over all possible IAs may benefit the VQ designer in estimating the performance of a given assignment. Given a VQ structure, upper and lower bounds on the "assignment gain" benefit the VQ designer searching for an efficient assignment. The evaluation of the average performance, over all IAs, can also help in revealing how well a given assignment performs.

The remainder of the correspondence is organized as follows. In Section II, the distortion due to channel errors is defined. The optimization of the channel distortion over all possible IAs is discussed. In Section III, bounds on the performance of a given VQ system under a given symmetric and memoryless channel, over all possible IAs, are obtained. Although the bounds are found to coincide with more general QAP bounds, the new derivation method allows us to compare these bounds with previously published VQ IA results [16]. A related expression for the average performance over all IAs is presented in Section IV. Special cases and numerical results, obtained in simulations, are presented and discussed in Section V, while conclusions are given in Section VI.

II. Channel Distortion

A vector quantization system is characterized by a set of codevectors and a corresponding partition of the signal space R of all possible input vectors $\underline{x}$. This space is partitioned into N regions, $R_i, i = 0, 1, \ldots, N-1$. These regions cover the whole signal space and are nonoverlapping

$$\bigcup_{i=0}^{N-1} R_i = R$$
$$R_i \cap R_j = \varnothing \quad i \neq j, \tag{4}$$

Each partition region R_i has a corresponding *reconstruction* (or *representation) vector* $\underline{\phi}_i$. For the special case of *centroid quantizers*

$$\underline{\phi}_i = \int_{R_i} \underline{x} \cdot p(\underline{x}) \cdot d\underline{x} / \int_{R_i} p(\underline{x}) \cdot d\underline{x}.$$

The encoder accumulates a K-dimensional vector of source samples $\underline{x}$. The symbol $y(t) = i$ is emitted if $\underline{x} \in R_i$. The corresponding channel symbol $z(t) = \Pi(i)$ is transmitted through the channel. The channel's output is a random mapping of this transmission. Upon receiving the channel symbol $\hat{z}(t) = j$, the decoder emits the reconstruction vector that corresponds to the index $\prod^{-1}(j)$.

The overall distortion of the VQ-based communication system is

$$\begin{aligned} D_T &= E[d(\underline{x}, \hat{\underline{x}})] \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \{\pi \cdot H \cdot \pi^T\}_{ij} \int_{R_i} d(\underline{x}, \underline{\phi}_j) \cdot p(\underline{x}) \cdot d\underline{x}. \end{aligned} \tag{5}$$

In (5), the permutation is represented by a permutation matrix π, whose entries are 0's and 1's and the sum of elements in each of its rows and columns is 1. The permutation matrix is self-orthogonal, i.e., $\pi\pi^T = I$.

For the perfect channel, $H = I$, the permutation matrix π is of no importance, and the only factor affecting system performance is the *quantization distortion—*D_Q

$$\begin{aligned} D_Q &= D_T|_{H=1} = E[d(\underline{x}, \hat{\underline{x}})]|_{H=1} \\ &= \sum_{i=0}^{N-1} \int_{R_i} d(\underline{x}, \underline{\phi}_i) \cdot p(\underline{x}) \cdot d(\underline{x}). \end{aligned} \tag{6}$$

In the following analysis, the *channel distortion* D_C, is defined by

$$\begin{aligned} D_C &= \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1} \{\pi \cdot H \cdot \pi^T\}_{ij} \cdot d(\underline{\phi}_i, \underline{\phi}_j) \\ &= \operatorname{trace}\{P \cdot \pi \cdot H \cdot \pi^T \cdot D\} \\ &= \operatorname{trace}\{D \cdot P \cdot \pi \cdot H \cdot \pi^T\} \end{aligned} \tag{7}$$

where p_i is the probability of

$$\underline{x} \in R_i, \quad p_i = \int_{R_i} p(\underline{x}) \cdot d\underline{x}.$$

The matrix P in (7) is a diagonal matrix, which contains these probabilities, i.e., $P = \operatorname{diag}\{p_0, p_1, \ldots, p_{N-1}\}$, and the entries of the matrix D are the distances between all possible pairs of reconstruction vectors $D_{ij} = d(\underline{\phi}_i, \underline{\phi}_j)$.

It is shown in [9], [11] that for the squared Euclidean distance measure and centroid quantizers, the overall distortion is the sum of the quantization and channel distortions: $D_T = D_Q + D_C$. This result is also applicable for quantizers with a large number of codevectors $(N \to \infty)$ [30].

For the special case of a uniform *scalar* quantizer, a uniform source, and transmission over a binary symmetric channel, it is shown in [28] that the NBC assignment, corresponding here to $\pi = I$ is an optimal assignment.

III. Performance Bounds

Minimization of channel distortion, as defined in (7), over all possible IAs (or permutation matrices π) is known to be a special case of the QAP [8]. The QAP is known to be NP-complete and therefore obtaining optimal assignments may not be feasible.

In this section, we introduce lower and upper bounds on the channel distortion D_C, under memoryless channels with a symmetric transition matrix, over all possible assignments (or permutation matrices—π). As in [5], [17], we define a symmetric matrix, *the weighted distance matrix*, $\hat{D}$ as

$$\hat{D} = DP + P^T D^T \tag{8}$$

so that by using the symmetry property of the channel matrix H, the channel distortion becomes

$$D_C = \frac{1}{2} \operatorname{trace}\{H \pi^T \hat{D} \pi\}. \tag{9}$$

The proposed bounding technique is based on eigenvalues arguments. Instead of optimizing over the (discrete) family of matrices covering all possible assignments π, we optimize over a wider (continuous) family.

A fundamental step in this optimization procedure is to replace the *weighted distance* matrix $\hat{D}$, defined in (8), by another symmetric matrix, the *balanced weighted distance—*$\tilde{D}$, such that the all-ones vector $\underline{1} = [1 \ 1 \ \cdots \ 1]^T$ is its eigenvector, while D_C is just changed by a known constant. This goal is achieved by the following procedure.

First, we denote a matrix of the form shown next as a "column structured" matrix

$$C_i = \begin{bmatrix} 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \end{bmatrix} = \underline{1} \cdot \underline{e}_i^T \tag{10}$$

$\uparrow$ ith column

where $\underline{e}_i^T = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]$ and the 1 is located at the ith location. Recalling that H represents probabilities, the sum of elements in any of its rows is one, so the vector $\underline{1}$ is an eigenvector of $H : H \cdot \underline{1} = \underline{1}$. The same argument is valid for the ith column of the matrix $C_i : H \cdot C_i = C_i, i = 0, 1, \ldots, N-1$.

We now construct a symmetric matrix $\alpha(C_i + C_i^T)$, where α is a scalar, denoted here as a *cross-structured matrix*. It is simple to show that, regardless of the permutation matrix π, adding a *cross-structured* matrix to the matrix $\hat{D}$ changes the expression in (9) just by the addition of the scalar α

$$\frac{1}{2} \operatorname{trace}\left\{H \pi^T \left[\hat{D} + \alpha\left(C_i^T + C_i\right)\right] \pi\right\} = \frac{1}{2} \operatorname{trace}\{H \pi^T \hat{D} \pi\} + \alpha \tag{11}$$

Let s_i denote the sum of the elements in the ith row of the matrix $\hat{D}$

$$s_i = \sum_{j=0}^{N-1} (\hat{D})_{ij} \tag{12}$$

and let $s_{\max} = \max_i s_i$ denote the largest sum.

In order to achieve the desired property $\tilde{D} \cdot \underline{1} = \omega_0 \underline{1}$, for some ω_0, all rows of $\tilde{D}$ must have the same sum of entries. Let us examine the effect of adding the "*cross-structured*" matrix $\alpha(C_i + C_i^T)$ to a general matrix M of size $N \times N$. The sum of elements in all rows except for the ith row is increased by α, while the sum of elements in the ith row is increased by $(N+1) \cdot \alpha$.

Therefore, we define the matrix $\tilde{D}$ to be

$$\tilde{D} = \hat{D} \sum_{i=0}^{N-1} \alpha_i (C_i + C_i^T) \tag{13}$$

where $\alpha_i = (s_{\max} - s_i)/N$. Having $s_{\max} \geq s_i$, the scalars α_i are all nonnegative. By adding N *cross-structured* matrices, some of which are all zeros, we get that $\tilde{D}$ is a symmetric matrix with all its rows having the same sum of elements, and with the desired property $\tilde{D} \cdot \underline{1} = \omega_0 \underline{1}$. We shall refer to $\tilde{D}$ as the *balanced weighted distance matrix*. The channel distortion can now be written as

$$D_C = \frac{1}{2} \mathrm{trace}\{H \pi^T \tilde{D} \pi\} - S \tag{14}$$

where $S = \sum_{i=0}^{N-1} \alpha_i$.

At this point, it is interesting to note that both the channel matrix H and the balanced weighted distance matrix $\tilde{D}$ are symmetric, have nonnegative entries, and have the vector $\underline{1} = [1\ 1\ \cdots\ 1]^T$ as an eigenvector. Moreover, because of the symmetry property, all eigenvalues of both matrices are real.

We note ([22, Sec. 15.7]) that the eigenvalue 1 of the matrix H and the eigenvalue $\omega_0 > 0$ of the matrix $\tilde{D}$, both corresponding to the eigenvector $\underline{1}$, are each the largest eigenvalue in absolute value of the corresponding matrix. (There may be negative eigenvalues, but smaller in absolute value).

Next, we use eigenvalue arguments to obtain bounds the channel distortion. We perform a unitary diagonalization on both matrices

$$\begin{aligned} H &= V \cdot \Lambda \cdot V^T, \qquad V \cdot V^T = I \\ \tilde{D} &= W \cdot \Omega \cdot W^T, \quad W \cdot W^T = I. \end{aligned} \tag{15}$$

Without loss of generality, we sort the eigenvalues (and corresponding eigenvectors) in Λ and Ω in decreasing order. Substituting (15) into (14)

$$\begin{aligned} D_C &= \frac{1}{2} \mathrm{trace}\{V \Lambda V^T \cdot \pi^T W \Omega W^T \pi\} - S \\ &= \frac{1}{2} \mathrm{trace}\{\Lambda \Psi \Omega \Psi^T\} - S \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \lambda_i \omega_j \psi_{ij}^2 - S \end{aligned} \tag{16}$$

where we define $\lambda_i = \Lambda_{ii}, \omega_i = \Omega_{ii}, i = 0, 1, \ldots, N-1$, and the matrix Ψ is defined as $\Psi = V^T \pi^T W$. The matrix Ψ is also unitary since $\Psi \Psi^T = V^T \pi^T W W^T \pi V = I$.

For the special case of an L-bit binary word transmitted through a BSC, the channel transition matrix H is known to be diagonalized by the Hadamard matrix and the eigenvalues $\lambda_i, i = 0, 1, \ldots, N-1$ are known explicitly [6], [7], [17]. There are $L+1$ distinct eigenvalues, $(1-2q)^m, m = 0, 1, \ldots, L$, each with multiplicity $\binom{L}{m}$, where q is the BER.

Observe that since the first column of both V and W is $\underline{v}_0 = \underline{w}_0 = (1/\sqrt{N})\underline{1}$, the sum of elements in the remaining columns of both matrices is zero. The structure of $\Psi = V^T \pi^T W$ is therefore

$$\Psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & ? & \\ 0 & & & \end{bmatrix} \tag{17}$$

where the question mark represents unknown entries.

In order to obtain upper and lower bounds over all possible IAs, we relax the constraint that the matrix Ψ in (16) equals to $V^T \pi^T W$ for some permutation matrix—π. The relaxation is done in two steps. In the first step, we replace the discrete family of matrices Ψ by the continuous family of unitary matrices having a general structure as in (17). In the second step, we replace the unitary requirement by a more relaxed condition. We merely demand that the sum of squares of the elements in each row and column is 1. We shall show that the second relaxation still results in a unitary matrix and hence does not degrade the tightness of the bounds.

In order to obtain the extreme values of the relaxed problem, we state the following optimization problem, using the property that the sum of squares of the elements in each row and column of a unitary matrix (Ψ in this case, with elements Ψ_{ij}) is equal to 1

$$\begin{aligned} &\min_{\Psi} / \max_{\Psi} \left(\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \lambda_i \omega_j \psi_{ij}^2 \right) \\ &\text{subject to} \sum_{i=1}^{N-1} \psi_{ij}^2 = 1, \qquad j = 1, 2, \ldots, N-1 \\ &\qquad\qquad \sum_{j=1}^{N-1} \psi_{ij}^2 = 1, \qquad i = 1, 2, \ldots, N-1. \end{aligned} \tag{18}$$

Note that the first row ($i = 0$) and the first column ($j = 0$) are independent of the permutation and were omitted from the optimization problem. We denote the solutions for the minimum/maximum problems by $\Psi_{\min} / \Psi_{\max}$, respectively. The solution of the optimization problems is given in [25]

$$\begin{aligned} &\text{Minimum value: } \sum_{i=1}^{N-1} \lambda_i \cdot \omega_{N-i} \\ &\text{Corresponding to:} \\ &\Psi_{\min} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & \pm 1 \\ \vdots & & & \pm 1 & \\ 0 & & \iddots & & \\ 0 & \pm 1 & & & 0 \end{bmatrix} \\ &\text{Maximum value: } \sum_{i=1}^{N-1} \lambda_i \cdot \omega_i \\ &\text{Corresponding to:} \\ &\Psi_{\max} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \pm 1 & & & 0 \\ \vdots & & \ddots & & \\ 0 & & & \pm 1 & \\ 0 & 0 & & & \pm 1 \end{bmatrix}. \end{aligned} \tag{19}$$

Note that the matrices $\Psi_{\min}$ and $\Psi_{\max}$ are unitary. This implies that the second relaxation did not worsen bounds tightness.

Applying these solutions to (18), the bounds on the channel distortion, over all possible IAs, are therefore

$$\frac{1}{2}\lambda_0 \omega_0 - S + \frac{1}{2} \sum_{i=1}^{N-1} \lambda_i \cdot \omega_{N-i} \leq D_C \leq \frac{1}{2}\lambda_0 \omega_0 - S + \frac{1}{2} \sum_{i=1}^{N-1} \lambda_i \cdot \omega_i. \tag{20}$$

It turns out that the upper and lower bounds obtained here coincide with the QAP results [25, Theorem 2.1 and 4.2, respectively]. How-

ever, the preceding derivation allows us in the sequel (Section V-A.2) to compare the QAP lower bound with a published lower bound.

Another representation of the bounds may be obtained by using the fact that $\lambda_0 = 1$ and, as may be seen from (13), $\omega_0 = S + s_k$:

$$\frac{1}{2N}\underline{1}^T \cdot \hat{D} \cdot \underline{1} + \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i \cdot \omega_{N-i} \le D_C \le \frac{1}{2N}\underline{1}^T \cdot \hat{D} \cdot \underline{1} + \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i \cdot \omega_i \quad (21)$$

The first part of both inequalities is independent of the channel. The second part depends on the eigenvalues of both the channel and the balanced weighted distance matrix.

In conclusion, in order to find the lower and upper bounds, given the channel transition matrix H, the VQ distance matrix D, and the *a priori* probabilities matrix P, one should carry out the following steps.

1. From D and P, calculate the scalars $s_i, i = 0, 1, \ldots, N-1$, using (12), $\alpha_i, i = 0, 1, \ldots, N-1, S$ (14), and the *weighted distance matrix*, $\tilde{D}$, using (13).
2. Calculate the eigenvalues of the *channel matrix* $H(\lambda_i, i = 0, 1, \ldots, N-1)$, and those of the *weighted distance matrix* $\tilde{D}(\omega_i, i = 0, 1, \ldots, N-1)$.
3. Calculate the upper and lower bounds using (20) or (21), where in the latter also (8) needs to be applied.

The upper and lower bounds in (20), (21), were obtained by using $2N$ constrains on the sum of squares of elements in each row and column of the unitary matrix $\Psi = V^T \pi^T W$. It is possible to add further linear constraints, thus achieving tighter upper and lower bounds. However, the solution turns out to be numerical rather than a closed analytical formula.

IV. Average Performance Over All Index Assignments

Having found lower and upper bounds on the channel distortion, it is also useful to obtain the average value of the channel distortion over all possible IAs. The average value can help in ranking a given assignment.

From (16), this average value is given by

$$\langle D_C \rangle = \frac{1}{2N!}\sum_{\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \lambda_i \omega_j \psi_{ij}^2 - S = \frac{1}{2N!}\sum_{\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \lambda_i \omega_j \left[\sum_{k=0}^{N-1} v_{ki} w_{\pi(k)j}\right]^2 - S \quad (22)$$

where the permutation is denoted by π and $v_{ki}, w_{\pi(k)j}$ are the elements of the matrices V and W, respectively. It may be shown that the ensemble average is

$$\langle D_C \rangle = \frac{1}{2}\lambda_0\omega_0 - S + \frac{1}{2(N-1)}\sum_{i=1}^{N-1}\lambda_i \sum_{j=1}^{N-1}\omega_j = \frac{1}{2N}\underline{1}^T \cdot \hat{D} \cdot \underline{1} + \frac{1}{2(N-1)}[\mathrm{trace}(H) - \lambda_0] \cdot [\mathrm{trace}(\tilde{D}) - \omega_0]. \quad (23)$$

Note that the average value in (23) corresponds to a matrix Ψ with the following structure:

$$\Psi_{\langle D_C \rangle} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta & \beta & \cdots & \beta \\ 0 & \beta & \beta & & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \beta & \beta & & \beta \end{bmatrix} \quad (24)$$

where $\beta = \pm(1)/(\sqrt{N-1})$.

The matrix $\Psi_{\langle D_C \rangle}$ in (24) is not unitary and therefore does not correspond to any valid permutation.

Comparing $\Psi_{\langle D_C \rangle}$ in (24) with the structure of the matrices $\Psi_{\min}$ and $\Psi_{\max}$ that correspond to the lower and upper bounds (19), respectively, one observes that the performance of a specific permutation corresponds to the geometric relations among the columns of the matrix V and the columns of the matrix $\pi^T W$. A permutation that aligns the column of the matrix V with the columns of the matrix $\pi^T W$, in direct (reverse) order, results in "poor" ("good") performance. A permutation that does not align the two sets of columns will typically results in "average" performance. This geometric interpretation can help in obtaining future suboptimal IA algorithms. A suboptimal algorithm may be based on a permutation matrix π in $\Psi = V^T \pi^T W$ that approximates $\Psi_{\min}$ in (19). Other geometric interpretations of the IA problem may be found in [15]–[17].

V. Special Cases and Simulation Results

In this section, we examine several special cases and compare the lower and upper bounds with the average distortion over all IA, as well as with the distortion of assignments that were obtained in simulations. We used the well-known index-switching algorithm [12], [13] to obtain "good" and "poor" IAs. According to this algorithm, after selecting an initial assignment, indices of codevectors are randomly exchanged. When searching for a good (poor) assignment, the new assignment is kept if it performs better (worse) than its predecessor.

A. Special Cases

1) A Uniform Scalar Quantizer and a Uniform Source Under the BSC: We obtain here the bounds in (20) for the case of a L-bit ($N = 2^L$ levels) uniform scalar quantizer, whose support is $[-1, 1]$, and a uniform source, operating under the BSC. For the mean-squared error distortion measure the resulting upper and lower bounds are

$$\frac{2(N-1)(N+1)}{3N^2}2q \le D_C \le \frac{2(N-1)(N+1)}{3N^2}[1-(1-2q)^L] \quad (25)$$

where q is the BER. It is interesting to see that for small values of q, the ratio between the upper and lower bounds is L.

The lower bound coincides with the performance of the NBC, which is an optimal assignment for this case, as shown in [28] and demonstrated in [3], [6], [7]. The worst IA for this case is given in [29], resulting in a channel distortion given by

$$D_{WIA} = q(1-2q)^{L-1} + \frac{2(N-1)(N+1)}{3N^2}[1-(1-2q)^{L-1}]. \quad (26)$$

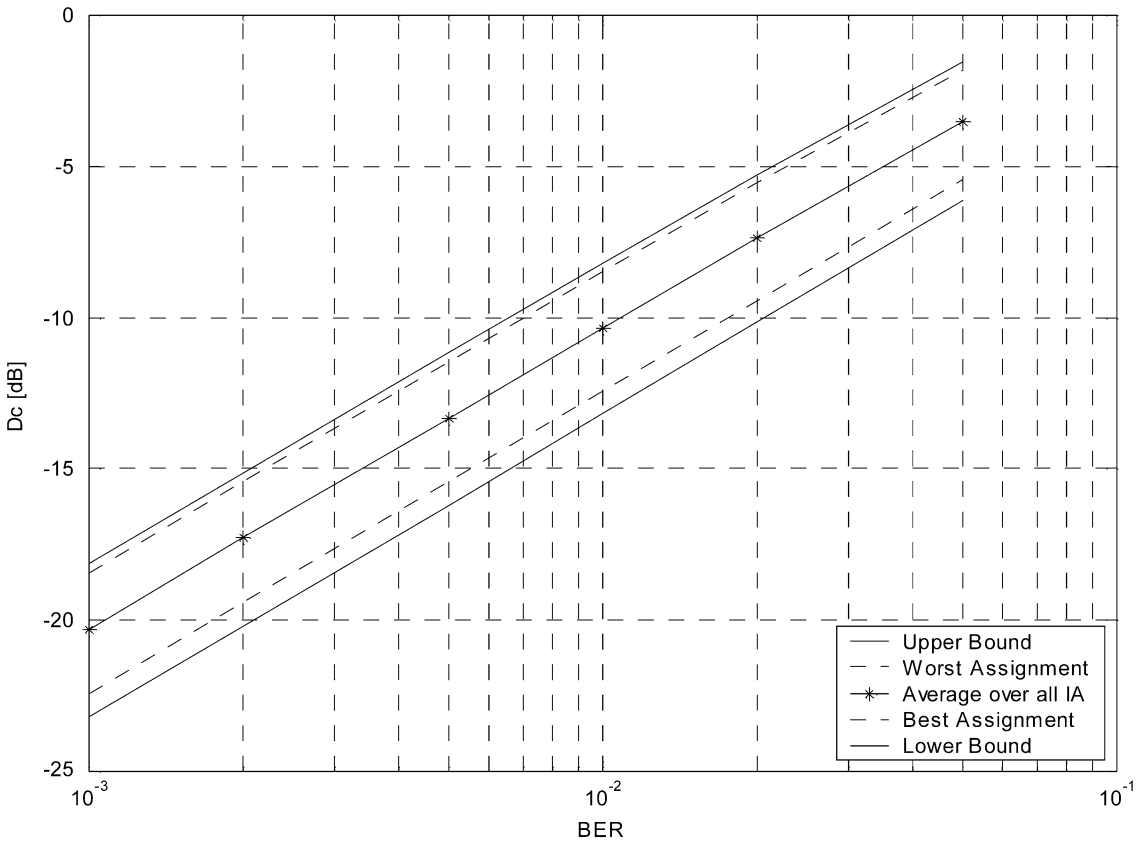

Fig. 2. Upper and lower bounds and the average channel distortion, over all possible IAs, for a 3-bit PDF-optimized scalar quantizer and a Gaussian source under the BSC. The bounds are compared with the best and worst assignments.

A numerical comparison between the performance of the worst IA and the upper bound (25) reveals that the upper bound is 0.25 dB higher for a 4-bit quantizer and only 0.13 dB higher for an 8-bit quantizer.

Using (22) for this case, the average distortion over all IAs is

$$\langle D_C \rangle = \frac{2(N+1)}{3N}[1-(1-q)^L]. \tag{27}$$

For small values of the BER, $q \to 0$, the ensemble average approaches zero linearly with q

$$\langle D_C \rangle \approx \frac{2L(N+1)}{3N}q, \qquad \text{as } q \to 0 \tag{28}$$

This expression agrees with an asymptotic result given in [9].

2) Maximum Entropy Vector Quantizers Under the BSC: For the special case of a maximum-entropy quantizer (equally probable quantization regions) with a quadratic distortion measure and the BSC, an asymptotic lower bound is given in [16]

$$D_C \geq \frac{4q}{N}\sum_{n=0}^{N-1}\phi_n^T\phi_n, \qquad \text{as } q \to 0 \tag{29}$$

where ϕ_n is the representation vector of the nth partition region. We define

$$Y = [\phi_0 \ \ \phi_1 \ \ \cdots \ \ \phi_{N-1}].$$

We also assume, without loss of generality, that $\sum_{n=0}^{N-1}\phi_n = 0$.

To compare (29) with the proposed lower bound, we note first that the channel matrix eigenvalues λ_i, $i = 0, 1, \ldots, N-1$ are $(1-2q)^m$, $m = 0, 1, \ldots, L$, each with multiplicity $\binom{L}{m}$. We represent these eigenvalues by $\lambda_i = (1-2q)^{m_i}, i = 0, 1, \ldots, N-1$, such that $m_0 = 0, m_1 = 1, \ldots, m_L = 1, m_{L+1} = 2, \ldots, m_{N-1} = L$. Note that the eigenvalues are sorted in descending order. In this case, the weighted distance matrix (13) is

$$\begin{aligned}\tilde{D} &= \frac{2}{N}\left[D + \sum_{i=0}^{N-1}\left(\phi_k^T\phi_k - \phi_i^T\phi_i\right)\cdot\left(C_i + C_i^T\right)\right] \\ &= \frac{2}{N}\left[\sum_{i=0}^{N-1}\phi_i^T\phi_i\cdot\left(C_i + C_i^T\right) - 2Y^TY \right. \\ &\quad \left. + \sum_{i=0}^{N-1}\left(\phi_k^T\phi_k - \phi_i^T\phi_i\right)\cdot\left(C_i + C_i^T\right)\right] \\ &= \frac{2}{N}\left[\phi_k^T\phi_k\cdot\sum_{i=0}^{N-1}\left(C_i + C_i^T\right) - 2Y^TY\right] \\ &= \frac{4\phi_k^T\phi_k}{N}\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix} - \frac{4}{N}Y^TY \end{aligned} \tag{30}$$

where k is selected such that $\phi_k^T\phi_k \geq \phi_i^T\phi_i$.

One may verify that the term S for (30) is given by

$$S = 2\phi_k^T\phi_k - (2)/(N)\sum_{i=0}^{N-1}\phi_i^T\phi_i.$$

The first eigenvalue of $\tilde{D}$ (corresponding to the eigenvector $\underline{w}_0 = (1/\sqrt{N})\underline{1}$) is $\omega_0 = 4\phi_k^T\phi_k$, and the sum of the eigenvalues of $\tilde{D}$ (sum of elements on the main diagonal) equals $2S$. The remaining $N-1$ eigenvectors are orthogonal to $\underline{1}$ and since Y^TY is positive semidefinite, they are all nonpositive, $\omega_i \leq 0, \forall i \neq 0$. Moreover, the sum of these remaining $N-1$ eigenvalues is therefore $(-4)/(N)\sum_{i=0}^{N-1}\phi_i^T\phi_i$.

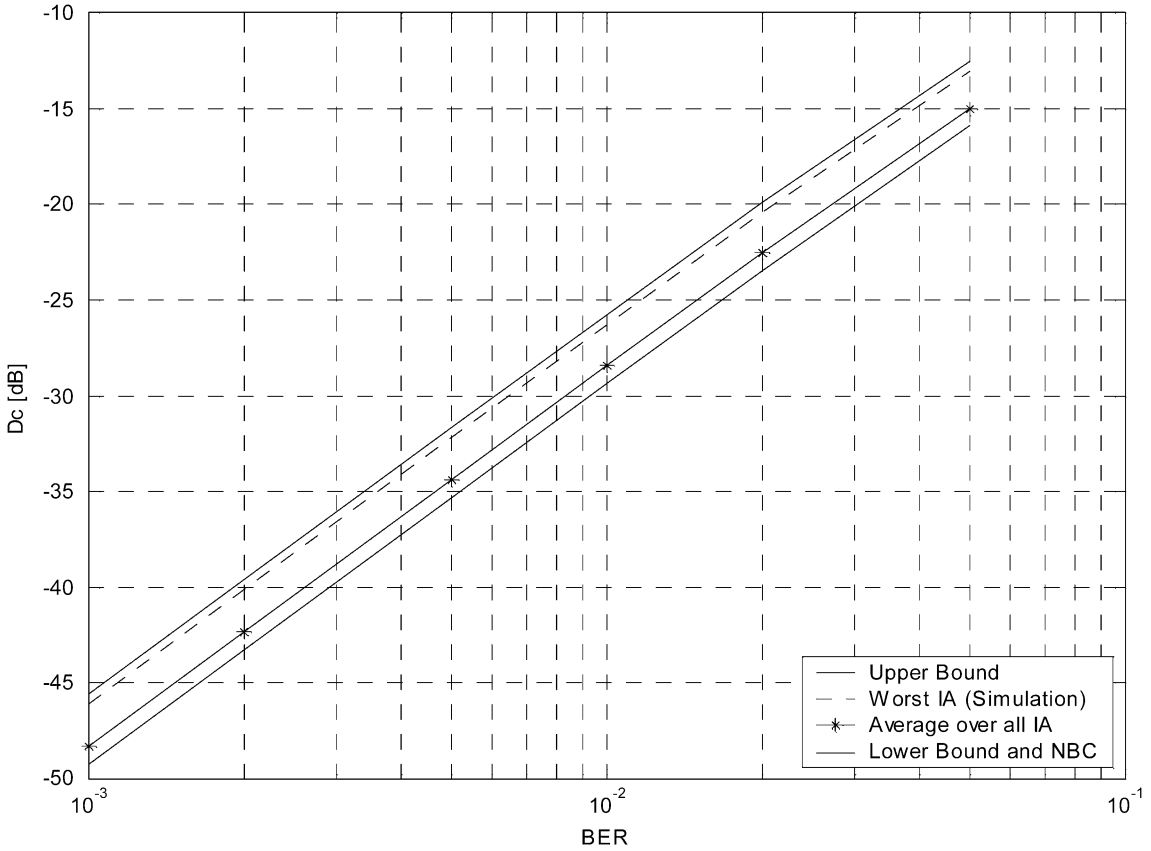

Fig. 3. Upper and lower bounds and the average channel distortion, over all possible IAs, of a 4-bit uniform scalar quantizer and a uniform source under the BSC with a $(7, 4)$Hamming error-correctng code. The upper bound is compared with a "poor" assignment obtained by simulations. The lower bound coincides with the performance of the NBC.

Using the properties $m_i \geq 1, \quad i = 1, 2, \ldots, N-1$ and $\omega_i \leq 0$, $\forall i \neq 0$, the corresponding lower bound is bounded by (for $0 \leq q \leq 0.5$)

$$\begin{aligned} &\frac{1}{2}\left(\omega_0 + \sum_{i=1}^{N-1}(1-2q)^{m_i}\cdot\omega_{N-1}\right) - S \\ &\geq \frac{1}{2}\left(\omega_0 + \sum_{i=1}^{N-1}(1-2q)\cdot\omega_{N-1}\right) - S \\ &= -q\sum_{i=1}^{N-1}(\omega_{N-i}) = \frac{4q}{N}\sum_{i=0}^{N-1}\phi_i^T\phi_i \end{aligned} \tag{31}$$

where equality hold only for the trivial cases of $q = 0$ or $L = 1$. Hence, for all cases of interest, the proposed lower bound is better than the bound in (29), give in [16].

B. Simulation Results

1) A 3-bit PDF Optimized Quantizer and a Gaussian Source Under the BSC: For 3-bit quantizers there are $8! = 40320$ possible assignments so that an exhaustive search is possible. We consider a 3-bit probability density function (PDF)-optimized scalar quantizer ([25, Ch. 4]), for a Gaussian source and a BSC. The bounds obtained from (20) are shown in Fig. 2. It can be seen that the slope of the lines is roughly 10 dB/decade, i.e., reducing the BER by a factor of 10 results in a 10-dB lower distortion. The channel distortion is approximately proportional to the BER. The upper bound is about 0.3 dB higher than channel distortion due to the worst possible IA. The lower bound is 0.8 dB lower than the distortion for the best assignment (found here by exhaustive search).

2) A 4-Bit Uniform Quantizer and a Uniform Source Using a $(7, 4)$ Hamming Error-Correcting Code Under the BSC: Consider a 4-bit uniform scalar quantizer and a uniform source. The digital information is sent through a BSC utilizing a $(7, 4)$ Hamming error-correcting code [26]. The channel matrix H is different from the BSC case. We examine a single entry of the channel transition matrix H in this case. Assume the encoder needs to transmit the index i. The corresponding Hamming codeword $c(i)$ (7 bits) is sent through the BSC. Each Hamming code decoder output $c(j)$ is a result of one of eight possible BSC outputs (Hamming code decoder input). These are the Hamming codeword $c(j)$ and codewords that are different from $c(j)$ by 1 bit only. Each entry of H is therefore a sum of eight probabilities

$$\begin{aligned} \{H\}_{i,j} &= \mathrm{Prob}\{\text{index } j \text{ received} \,|\, \text{index } i \text{ transmitted}\} \\ &= \sum_{d_H(k,c(j))\leq 1} \mathrm{Prob}\{k \text{ received} \,|\, \text{codeword } c(i) \text{ transmitted}\} \end{aligned} \tag{32}$$

The bounds are shown in Fig. 3.

It can be seen that the slope of the graphs is 20 dB/decade, i.e., reducing the BER by a factor of 10 results in a 20-dB lower distortion. The channel distortion is approximately proportional to the square of the BER. The upper bound is about 0.5 dB away from the worst random assignment (out of 10^6 IAs) found in simulations. The proposed lower bound coincides with the performance of the NBC. The optimality of the NBC for this case was presented in [33]. The ratio between the upper and lower bounds is approximately 3.6 dB, compared with 6 dB for the BSC without channel protection. The implementation of the channel protection brought the bounds closer together, decreasing the effect of IA.

3) Two-Dimensional PDF-Optimized Vector Quantizer for a Gauss–Markov Source Under the BSC: We examine here a set of

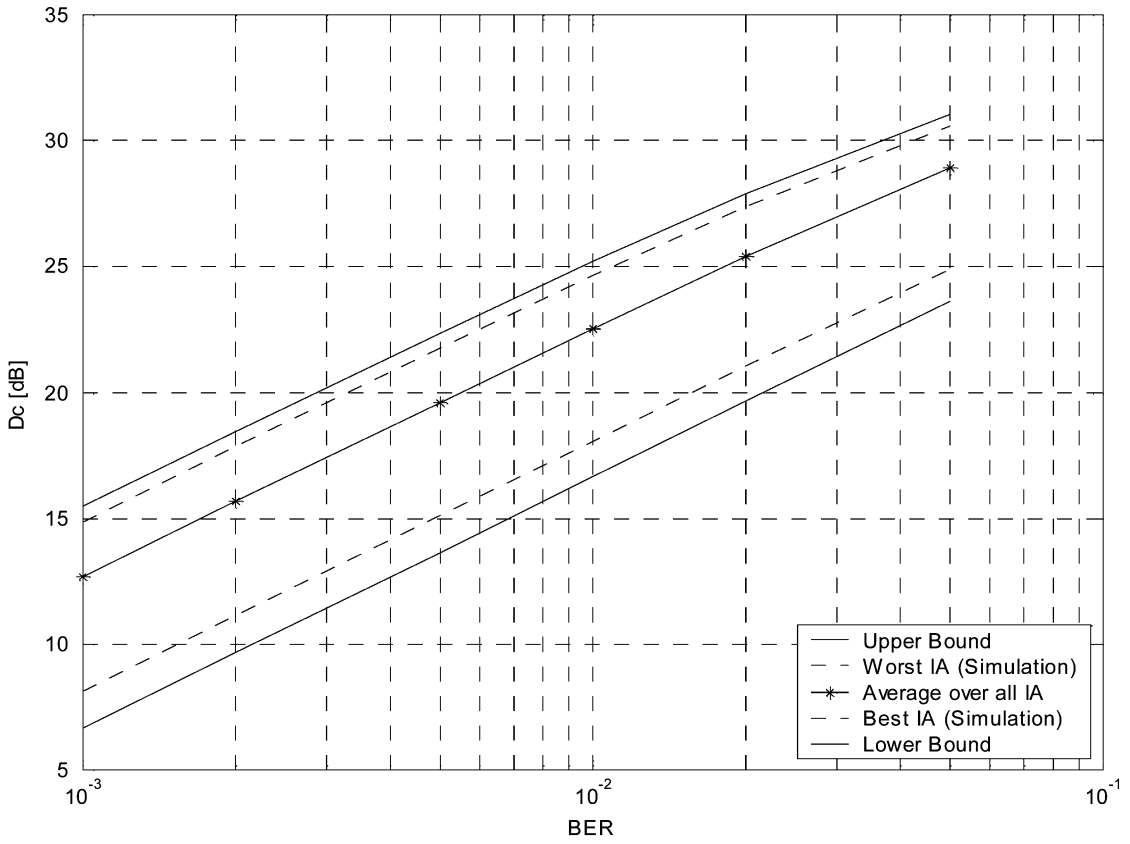

Fig. 4. Upper and lower bounds and the average channel distortion, over all possible IAs, of an 8-bit L*a*b*-space image vector quantizer under the BSC. The bounds are compared with "good" and "poor" assignments attained in simulations.

two-dimensional vector quantizers, designed for a Gauss–Markov source, with correlation $\rho = 0.5$, and different sizes. The vector quantizers were designed using the well-known LBG algorithm [1], [27]. The digital information is sent through a BSC.

The results obtained are shown in Table I.

It can be seen that the distance between upper and lower bounds increases with the complexity of the quantizer. The gap between the upper bound and the worst assignment found in simulations as well as the gap between the lower bound and the best assignment found in simulations (out of 10^6 IAs) also expand with VQ size.

Unfortunately, due to the huge amount of possible assignments and the suboptimality of the index-switching algorithm, we cannot state at this point if these gaps are due to an inadequate index-searching mechanism or as a result of insufficient bound tightness (or both). As seen earlier, the error-correcting code brought the bounds closer together. When an error-correcting code is applied, the relatively small ratio between the average distortion and the lower bound suggests that in this case one can resort to just choosing the best of several random assignments.

4) Three-Dimensional 8-Bit PDF-Optimized Vector Quantizer for Quantizing Images in the L∗a∗b∗ Color Space: The L*a*b* color space was developed by the CIE [31] in order to better match color representation to human color perception. Pixel colors are organized in three components: an achromatic (luminance) component L*, and two chromatic ones: a* and b*. Because color difference perception over the L*a*b* color space is approximately uniform, the squared Euclidean distance is considered to be an appropriate distortion measure in this color space. The nonlinear transformations between RGB and L*a*b* spaces can be found in [31]. We examine here an 8-bit, $N = 256$, vector quantizer from [32]. The computed bounds are shown in Fig. 4.

TABLE I
BOUNDS CHARACTERISTICS FOR TWO-DIMENSIONAL VECTOR QUANTIZERS DESIGNED FOR A GAUSS–MARKOV SOURCE, WITH CORRELATION $\rho = 0.5$. (NOA—NUMBER OF POSSIBLE ASSIGNEMNTS, UB/LB—UPPER/LOWER BOUND, AD—AVERAGE DISTORTION)

VQ size (bits)	4	4	6	8
NOA (power of 10)	13	13	89	506
Error correcting		(7,4) Hamm.		
UB/LB (dB)	3.6	2.0	5.7	7.2
UB gap (dB)	0.6	0.3	0.7	0.9
LB gap (dB)	0.6	0.2	1.3	2.0
AD/LB (dB)	2.0	0.7	3.6	4.8

The upper bound is about 0.6 dB higher than the channel distortion due to the worst IA obtained in the simulations (out of 10^6 IAs). The lower bound is 1.5 dB lower than the distortion for the best assignment obtained in simulations. The ratio between the upper and lower bounds is 8.8 dB, suggesting that a significant performance gain can be achieved by a good IA.

VI. CONCLUSION

In this correspondence, we have presented upper and lower bounds (and a related expression for the average performance) of the distortion due to channel errors for vector quantizers operating under channel errors, over all possible IAs. These bounds are based eigenvalues arguments and coincide with g QAP problem bounds.

The proposed bounding technique enables a comparison of the QAP lower bound to a previously published lower bound that was derived

for maximum entropy quantizers. The QAP lower bound is proved to be tighter.

The bounds enable the VQ designer to estimate the gain that may be obtained by a search for an efficient IA. Together with the average performance, the designer may evaluate the performance of a given IA.

Analytical and numerical examples were given for the BSC, with and without error correction. For 3-bit (8-level) quantizers, the bounds were compared with the best and worst assignment using an exhaustive search. For 4-bit and higher resolution quantizers, the bounds were compared with "good" and "poor" assignments obtained in simulations using a suboptimal index-switching algorithm.

For low- and intermediate-size vector quantizers, under the BSC, the bounds are reasonably close to the performance of the assignments found in simulations.

For large size VQs, there is a larger gap between the bounds and the simulation results. The number of possible IAs $(N!)$ does not allow an exhaustive search while the suboptimality of the index-switching algorithm leave the tightness issue of the proposed bounds to further study.

Utilization of error correction decreases the gap between the lower and the upper bounds, and both bounds are tighter. This result agrees with the intuition that channel protection reduces the importance of IA. Nevertheless, a good IA improves the quantization system performance and does not add complexity to its implementation.

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REFERENCES

[1] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Boston, MA: Kluwer Academic, 1991.

[2] G. Ben-David and D. Malah, "On the performance of a vector-quantizer under channel errors," in *Signal Processing VI Theories and Applications, Proc. EUSIPCO-92*. Amsterdam, The Netherlands: Elsevier, 1992, pp. 1685–1688.

[3] ——, "Properties of the natural binary code assignment for uniform scalar quantizers under channel errors," in *Proc. ECCTD-93*. Amsterdam, The Netherlands: Elsevier, 1993, pp. 773–778.

[4] ——, "Simple adaptation of vector-quantizers to combat channel-errors," in *Proc. 6th IEEE DSP Workshop*. Piscataway, NJ: IEEE Press, 1994, pp. 41–44.

[5] G. Ben-David, "Vector and scalar quantization under channel errors," Ph.D. dissertation, Technion-IIT, Israel, Jun. 1995. In Hebrew.

[6] S. W. McLaughlin, D. L. Neuhoff, and J. Ashley, "The optimality of the natural binary code," in *Proc. Joint DIMACS/IEEE Workshop on Coding and Quantization*, Piscataway, NJ, Oct. 1992, pp. 95–102.

[7] ——, "Optimal binary index assignments for a class of equiprobable scalar and vector quantizers," *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pp. 2031–2037, Nov. 1995.

[8] N. Farvardin and V. Vaishampayan, "Optimal quantizer design for noisy channels: An approach to combined source-channel coding," *IEEE Trans. Inf. Theory*, no. 6, pp. 155–159, Nov. 1987.

[9] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Trans. Inf. Theory*, vol. 36, no. 4, pp. 799–809, Jul. 1990.

[10] P. M. Pardalos, F. Rendl, and H. Wolkowicz, "The quadratic assignment problem: A survey and recent development," in *DIMACS Series in Discrete Mathematics and Theoretical ComputerScience*. Providence, RI: Amer. Math. Soc., 1994, vol. 16, pp. 1–42.

[11] A. Gersho and K. A. Zeger, "Vector quantizer design for memoryless noisy channels," in *Proc. IEEE Int. Conf. Communications*, Philadelphia, PA, Jun. 1988, pp. 1593–1597.

[12] ——, "Pseudo-Gray coding," *IEEE Trans. Commun.*, vol. 38, no. 12, pp. 2147–2158, Dec. 1990.

[13] J. R. B. De-Marca and N. S. Jayant, "An algorithm for assigning binary indices to the codevectors of a multi-dimensional quantizer," in *Proc. IEEE Intl. Conf. Communications*, Seattle, WA, Jun. 1987, pp. 1128–1132.

[14] P. Knagenhjelm, "A recursive design method for robust vector quantization," in *Proc. Int. Conf. Signal Processing Applications and Technology (Boston-92)*, Boston, MA, Nov. 1992, pp. 948–954.

[15] ——, "How good is your assignment?," in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing (ICASSP 93)*, Minneapolis, MN, Apr. 1993, pp. 423–426.

[16] P. Knagenhjelm and E. Agrell, "The Hadamard transform—A tool for index assignment," *IEEE Trans. Inf. Theory*, vol. 42, no. 4, pp. 1139–1151, Jul. 1996.

[17] M. Skoglund, "On channel-constrained vector quantization and index assignment for discrete memoryless channels," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2615–2622, Nov. 1999.

[18] P. C. Gilmore, "Optimal and suboptimal algorithms for the quadratic assignment problem," *J. Soc. Indust. Appl. Math*, vol. 10, no. 2, pp. 305–313, Jun. 1962.

[19] K. Hall, "An r-dimensional quadratic assignment placement algorithm," *Manag. Sci.*, vol. 77, no. 3, pp. 219–229, Nov. 1970.

[20] N. L. C. Talbot and G. C. Calway, "A fast index assignment method for robust vector quantization of image data," in *Proc. 1997 Int. Conf. Image Processing (ICIP'97)*, Washington, DC, Oct. 1997, pp. 674–680.

[21] J.-S. Pan, C.-S. Shieh, and S.-C. Chu, "Comparison study on VQ codevector index assignment," in *Proc. 5th Int. Conf. Spoken Language Processing (ICSLP'98)*, Sydney, Australia, 1998, pp. Th4Q1–Th4Q4. Paper 0031.

[22] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*. Orlando, FL: Academic, 1985.

[23] G. Finke, R. E. Burkard, and F. Rendl, "Quadratic assignment problems," *Ann. Discr. Math.*, vol. 31, pp. 61–82, 1987.

[24] F. Rendl and H. Wolkowicz, "Applications of parametric programming and eigenvalue maximization to the quadratic assignment problem," *Math. Progr.*, vol. 53, pp. 63–78, 1992.

[25] S. W. Hadley, F. Rendl, and H. Wolkowicz, "A new lower bound via projection for the quadratic assignment problem," *Math. Oper. Res.*, vol. 17, no. 3, pp. 727–739, 1992.

[26] H. Taub and D. L. Schilling, *Principles of Communication Systems*, 2nd ed. New-York: McGraw-Hill, 1986.

[27] N. S. Jayant and P. Noll, *Digital Coding of Waveforms*. Englewood Cliffs, NJ: Prentice-Hall, 1984.

[28] T. R. Crimmins, H. M. Horwitz, C. J. Palermo, and R. V. Palermo, "Minimization of mean-square error for data transmitted via group codes," *IEEE Trans. Inf. Theory*, vol. IT-15, no. 1, pp. 72–78, Jan. 1969.

[29] A. Méhes and K. Zeger, "Randomly chosen index assignments are asymptotically bad for uniform sources," *IEEE Trans. Inf. Theory*, vol. 45, no. 2, pp. 788–794, Mar. 1999.

[30] K. Zeger and V. Manzella, "Asymptotic bounds on optimal noisy channel quantization via random coding," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 1926–1938, Nov. 1994.

[31] Central Bureau of the Commission Internationale de L'Eclairage (CIE), 1986. Colometry, CIE publication no. 15.2.

[32] A. Spira, "Image paletization and compression for color limited displays," M.Sc. thesis, Technion–Israel Instite of Technology, Haifa, Israel, Jul. 2000.

[33] J. Ashley, *Error Correcting Codes and Quantization*. Yorktown Heights, NY: IBM Res. Div. , Jan. 1993. Internal publication.