Quadtree and Bit-Plane Decompositions as Particular Cases of the Generalized Morphological Skeleton*

R. Kresch and D. Malah

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel. E-mail: kresch@dekel.technion.ac.il

ABSTRACT — **In recent years, the Morphological Skeleton Decomposition of images had its framework extended, leading to a Generalized Skeleton Decomposition. This work shows how the Quadtree Decomposition of binary images and the Bit-Plane decomposition of Grayscale pictures can both be seen as particular cases of such a generalized Skeleton.**

Apart from its theoretical relevance, in relating image representations thought previously to be unrelated, the new results make it possible to obtain *multi-parameter* generalizations of the Quadtree and the Bit-Plane decompositions, supported by the generalized framework of Skeleton Decomposition.

1 Introduction

Originally, the Morphological Skeleton Decomposition was defined for *sets* in $\mathcal{R}^2$ as the collection of all the *maximal discs* contained in the set. On the other hand, the Quadtree Decomposition of a square-shaped binary image is obtained by recursevily splitting the image into four (usually equal) square sub-images, and checking their homogeneity before further splitting.

By considering the above conventional descriptions, one can hardly see any strong relationship between the two decomposition methods.

However, recent generalizations extended the scope of the Skeleton Decomposition [4, 7]. The Generalized Skeleton Representation is still defined as the collection of centers and radii of maximal elements, but the notions of “center”, “radius” and “element” are generalized. In this context, the Quadtree representation is shown to be a particular case, and therefore is subject to the same formulæ and have the same properties as the Generalized Skeleton. The same conclusion is shown to apply to the Bit-Plane Decomposition of grayscale images as well.

*This work was supported by the fund For the Promotion of Research at the Technion

2 Background on Morphological Skeleton and its Generalizations

The authors suppose the readers to be familiar with the basic morphological operations (dilation, erosion, opening and closing), both in the Euclidean translation-invariant case (denoted respectively as $\oplus$, $\ominus$, $\circ$, and $\bullet$ - see [1] for background material), and in the Lattice generical case (denoted respectively as δ, ε, γ, and ϕ - see [2]).

2.1 Original Skeleton Definition

Since its introduction, the Morphological Skeleton Decomposition was generalized several times. As mentioned above, it was originally defined (see [1]) for sets in $\mathcal{R}^2$ (continuous binary images), as the collection of centers and radii of the *maximal discs* inscribable in the given set. A maximal disc is a disc not contained in any bigger disc inscribable in the given set.

In one of the earliest, and better known, generalizations [3], the family of discs of increasing radii used in the decomposition was replaced by versions with increasing sizes of a pre-defined arbitrary convex shape B (e.g., a square or a rhombus). In addition, that new Skeleton was defined also for discrete binary images, i.e., sets in $\mathcal{Z}^2$. The discrete Skeleton subsets S_n $(n = 0, 1, \ldots)$, each containing the centers of the maximal elements of size n, can be calculated by the following discrete version of Lantuéjoul's formula [3]:

$$S_n = X \ominus nB - (X \ominus nB) \circ B. \qquad n = 0, 1, \ldots \tag{1}$$

where X is the original shape.

2.2 Skeleton in Boolean Lattices

Later on, the Morphological Skeleton was generalized further in [2], basically in two ways.

First, instead of applying to Euclidean Spaces only, it was applied to any *Boolean Lattice*. A Boolean Lattice $\mathcal{P}(E)$ is the collection of all subsets of a given set of "points" E, along with the *union* and *intersection* operations (see [2]). In the previous cases, E was equal to either $\mathcal{R}^2$ or to $\mathcal{Z}^2$.

As for the second generalization, instead of using structuring-elements, structuring functions (Variable Structuring-Elements [4]) are used. A structuring-function is any map relating each *point* in E to a *collection of points* in the Boolean Lattice $\mathcal{P}(E)$. Every structuring-function automatically defines a dilation $\delta(\cdot)$, an erosion $\varepsilon(\cdot)$, an opening $\gamma(\cdot)$ and a closing $\phi(\cdot)$ in the Boolean Lattice (see [2]).

In the *discrete* case, the generalized Lantuéjoul's formula can be written in the following way:

$$\begin{array}{c} S_n = \varepsilon^n(X) - \gamma\varepsilon^n(X). \\ n = 0, 1, \ldots \end{array} \tag{2}$$

The above formula decomposes a set X in the given Boolean Lattice into maximal elements of the form $\delta^n(x)$, $n = 0, 1, \ldots$, $x \in E$.

The Skeleton decomposition according to equation (2) is completely determined once the *Boolean Lattice* and the *structuring-function* are defined. In section 3 bellow, we show that the Quadtree and the Bit-Plane decompositions are particular cases of the above Skeleton, by selecting suitable Lattices and structuring-functions.

2.3 Multi-Parameter Skeletons

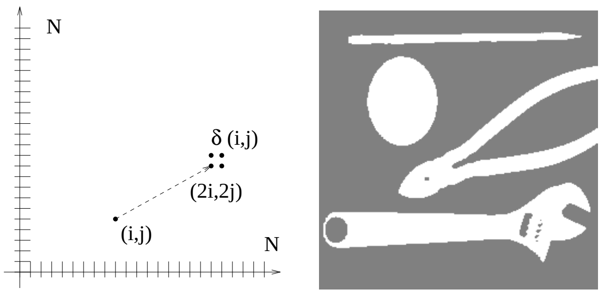
Equation (2), and the whole Skeleton scope, were further extended in [4] and in [7]. These generalizations include in the general Skeleton framework the notions of "variable-step decomposition" (as in the Generalized-Step Skeleton [5]) and "multi-parameter decomposition" (as in the Multi-Structuring-Element Skeleton [6]).

In [7], for example, a Multi-Parameter Skeleton was defined. For the simpler case of 2 discrete parameters, its subsets are given by:

$$\begin{array}{c} S_{n,m} = \varepsilon_1^n \varepsilon_2^m(X) - \bigcup_{i=1}^{2} \gamma_i \varepsilon_1^n \varepsilon_2^m(X), \\ (n, m) \in \mathcal{N}^2 \end{array} \tag{3}$$

where, now, the erosions and openings are obtained from 2 different structuring-functions $\delta_1(\cdot)$ and $\delta_2(\cdot)$. In this case, the given set X is decomposed into maximal elements of the form $\delta_1^n \delta_2^m(x)$, $(n, m) \in \mathcal{N}^2$, $x \in E$.

The above generalization apply also to the Quadtree and the Bit-Plane decompositions, since they are shown here to be particular cases of



(a) (b)

Figure 1: (a) The structuring-function for the Quadtree Decomposition. (b) Original binary 256×256-pixel image.

Skeleton. These results are specifically presented in section 4.

3 Quadtree and Bit-Plane Decompositions as Particular Cases

3.1 Quadtree Decomposition

Let us select the Boolean Lattice as the set $\mathcal{P}(\mathcal{N}^2)$ of all the subsets of $\mathcal{N}^2$, where $\mathcal{N}$ is the set of natural numbers, and let the structuring-function $\delta : \mathcal{N}^2 \to \mathcal{P}(\mathcal{N}^2)$ be (as depicted in Figure 1(a)) defined by:

$$\begin{array}{rcl} \delta(i,j) & = & \{(2i, 2j), (2i+1, 2j), \\ & & (2i, 2j+1), (2i+1, 2j+1)\}, \\ & & (i,j) \in \mathcal{N}^2 \end{array} \tag{4}$$

The *dilation* $\delta(X)$ derived from the above structuring function is equivalent to an *interpolation* process; it first upsamples the input binary image X, and then fills the gaps between samples by dilating it (in the translation-invariant sense, i.e., performing $\oplus$) by a 2×2-pixel squared structuring element. Therefore, the adjoint *erosion* $\varepsilon(X)$ is equivalent to a *decimation* process, where X is first eroded (in the translation-invariant sense - $\ominus$) by the same 2×2 pixel structuring element as above, and then downsampled. The related *opening* $\gamma(X)$ is the result of "decimation" followed by "interpolation".

The decomposition of a binary image, using (2) with the above selections, is the Quadtree decomposition *of the foreground* of the image. For example, Figure 1(b) shows a binary image, and Figure 2(a) shows its Skeleton subsets S_n, $n = 0, 1, \ldots$. The black pixels are the Skeleton Points; each representing a maximal element. In this case, maximal elements are squares of sizes $2^n \times 2^n$ pixels. Figure 2(b) shows the maximal elements for the above image.

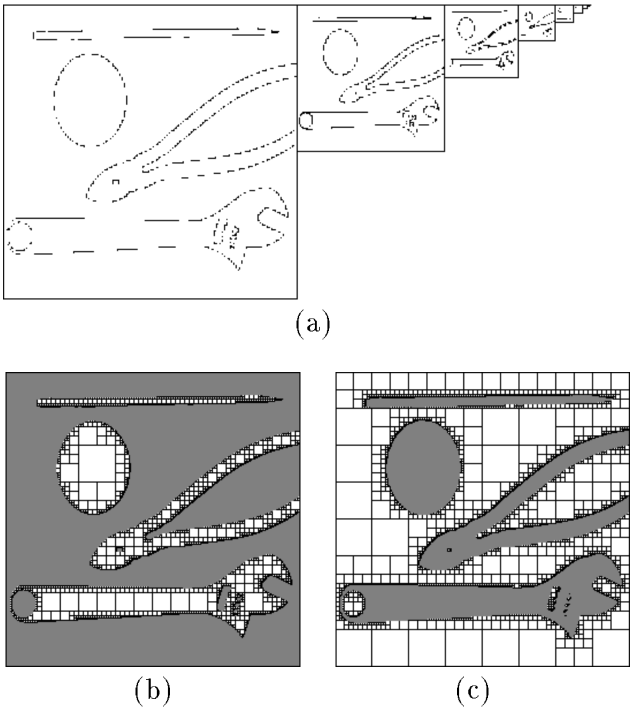


Figure 2: Quadtree decomposition as a Skeleton. (a) Skeleton Subsets S_n, $n = 0, 1, \ldots$, of the foreground decomposition, (b) Maximal Elements (foreground decomposition), (c) Maximal Elements (background decomposition).

By inverting each pixel of the image (interchanging foreground and background) and applying the above decomposition again, one obtains the Quadtree decomposition *of the background.* The corresponding maximal elements are shown in Figure 2(c). Full Quadtree representation consists of both foreground and background decompositions.

3.2 Bit-Plane Decomposition

For a Bit-Plane decomposition, we select the Lattice of *grayscale functions* complemented with ∞, i.e., functions with values in the range $\{0, 1, \ldots, 255\} \cup \{\infty\}$.

Before we proceed, we should remark that Lattices of *functions* are not *Boolean* Lattices [2]. Therefore, one can not define, in a rigorous way, *any* type of Skeleton Decomposition for *functions*. Nevertheless, it is common to apply Lantuéjoul's Formula (1), or its generalizations (2) and (3), also for functions, producing Skeleton-like function decompositions. Although not satisfying *all* the Skeleton properties, they do satisfy most of the important ones. The Bit-Plane decomposition is shown here to be a particular case of the above.

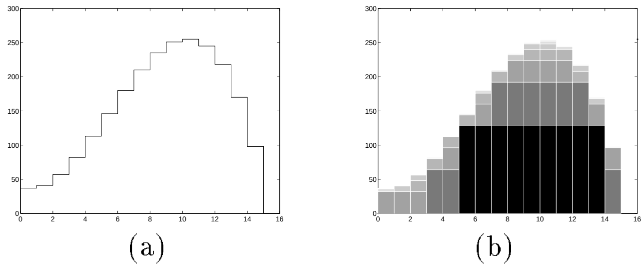


Figure 3: Bit-Plane Decomposition as a grayscale Skeleton. (a) Original discrete function, (b) decomposition into "Maximal Elements" (the rectangles); each "Maximal Element" represents a "1" in the appropriate Bit-Plane.

We select the following dilation $\delta(f)$ in the above Lattice (f being a function):

$$\delta(f(x)) = \begin{cases} 2f(x), & 0 \leq f(x) < 128 \\ \infty, & f(x) \geq 128 \end{cases} \qquad (5)$$

The corresponding erosion is $\varepsilon(f) = \lfloor f/2 \rfloor$, and the opening is $\gamma(f) = 2\lfloor f/2 \rfloor$. The sets S_n obtained by equation (2) in this case, with X replaced by f, are the Bit Planes of f.

For example, Figure 3(a) shows an one-dimensional discrete function. Figure 3(b) shows the result of the decomposition, where each "rectangle" represents a "1" (Skeleton Point) in the appropriate Bit-Plane (Skeleton subset). Although maximal elements can not be defined in this case, the above "rectangles" offer a suitable approximation.

4 Proposed Generalizations

4.1 Generalized Quadtree Decomposition

We define a 2-parameter Quadtree decomposition, by using equation (3) with the following structuring-functions:

$$\delta_1(i, j) = \{(i, 2j), (i, 2j + 1)\} \qquad (6)$$

$$\delta_2(i, j) = \{(2i, j), (2i + 1, j)\}. \qquad (7)$$

The above structuring functions are depicted in Figure 4.

The above Generalized Quadtree decomposes a binary image into *rectangles* rather than *squares* (see Figure 5).

As opposed to the original Quadtree decomposition, where the decomposition squares are disjoint, the decomposition rectangles of the Generalized Quadtree may overlap (see the light grey areas in Figures 5(b) and 5(c)). On the other hand, as opposed to standard Skeleton decompositions, where some Skeleton points are redundant

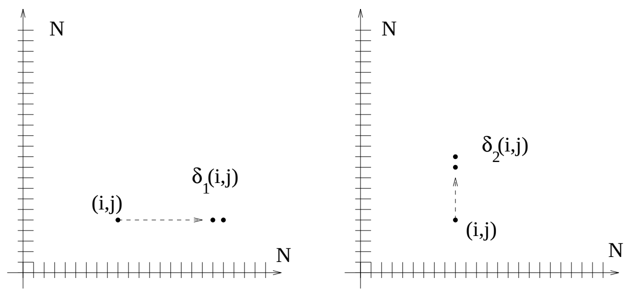


Figure 4: Structuring functions for the proposed Generalized Quadtree Decomposition.

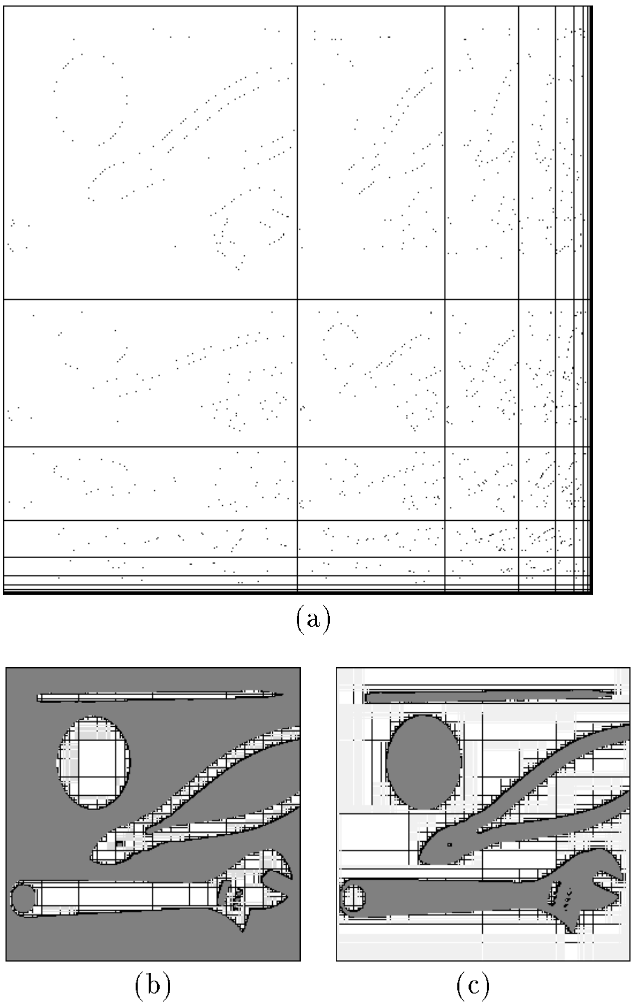


Figure 5: Proposed Generalized Quadtree decomposition. (a) Generalized Skeleton Subsets $S_{n,m}$, $(n, m) \in \mathcal{N}^2$, for the foreground decomposition, (c) Maximal Elements (foreground decomposition). The light grey areas represent overlapping of the decomposition rectangles, (d) Maximal Elements (background decomposition).

and may be removed, there are no redundant rectangles. The number of decomposition rectangles is usually much smaller than the number of decomposition squares, as can be seen when comparing Figure 2(a) with Figure 5(a). This has potential use in coding.

4.2 Bit-Plane–Quadtree Decomposition

A *Bit-Plane–Quadtree Decomposition* can be defined, by using Lantuéjoul's generalized equation (3) with δ_1 selected to be the "Bit-Plane dilation" defined in (5), and δ_2 selected to be the "Quadtree dilation" ("interpolation") defined in section 3.1, adapted to functions rather than binary images.

This provides a decomposition of a *grayscale* image into squares of size 2^n and graylevel 2^m each, where n and m are integers. The number of such squares is usually much smaller that the number of "1"s in the Bit-Plane decomposition. This also could be of potential use in coding.

5 Conclusion

We have shown that the Quadtree and the Bit-Plane decompositions can be seen as particular cases in the Generalized Skeleton Framework. Based on a further generalization of this framework, we were able to present a *Generalized Quadtree Decomposition*, and briefly describe a *Bit-Plane–Quadtree Decomposition*. Both have potential use in coding, since the number of their decomposition elements are usually much smaller than those of the standard decompositions.

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