Skeleton Redundancy Reduction Based on a Generalization of Convexity

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Abstract. We present a generalization of the concept of Convex Sets, based on the Morphological Closing operation, and study some of its properties. We also define Extreme Points of such Generalized Convex Sets, which generalize the notion of Extreme Points of Convex Sets.
Moreover, we apply the above notions to skeleton redundancy removal, and present an algorithm for obtaining an Error-Free Skeleton representation with reduced amount of redundant points, using morphological operations only.

1. Introduction

The concept of *Convexity* is of great interest in several areas, such as Shape Analysis, Pattern Recognition, Image Decomposition, and others. Many properties and relations concerning Convex Sets have been extensively studied and analyzed, and a number of generalizations of Convexity were proposed before (see [1] for example), in order to extend some of these properties and relations to sets which are not strictly convex. The latter is the purpose of this work too.

The generalization proposed here is based on the *Morphological Closing* (one of the four Morphological basic operations). The proximity between the *Convex-Hull* operator and the *Closing* operator has already been pointed out in [2], but a close look at the structure of their definitions shows that there is more than a proximity; the *Convex-Hull is actually a particular case of Closing.* Therefore, some important properties of the former operator are naturally extended for the latter one.

In section 2, we remind the definition and some properties of Convex Sets and present the proposed generalization, based on the Closing operator.

In Convex Sets, one can find *Extreme Points*, which have several properties. Among those properties, there is the ability of fully representing the whole set (if it is bounded), i.e., one can recover a bounded Convex Set from its Extreme Points only (this is done by simply applying the Convex-Hull operator). In section 3, we define Extreme Points for the previously generalized Convex Sets, and present *Morphological formulæ* for calculating them. We also study conditions for perfect reconstruction of the generalized Convex Sets from their Extreme Points.

Section 4 presents an application of the proposed generalization for *reducing the redundancy in Skeleton representations.* It consists of an *entirely morphological* algorithm for removing most of the redundant points of a given Skeleton.

Throughout the paper, we use the following notation: For A and B sets in $\mathcal{R}^2$, $A \oplus B$, $A \ominus B^s$ and $A \bullet B$ are, respectively, the Morphological Dilation, Erosion and Closing of A by B; $B^c = \{b \in \mathcal{R}^2 \mid b \notin B\}$ is the complement of B, $B^s = \{-b \mid b \in B\}$ is the transposed set of B, and $B_z = \{b+z \mid b \in B\}$ is the translation of B to the point $z \in \mathcal{R}^2$.

2. Convex Sets and Proposed Generalization

There are several acceptable definitions for Convex-Hull and Convex Sets. They are all equivalent, up to topological differences concerning the points on the boundary of the shapes. We can also define first the Convex-Hull and then use this definition for defining Convex Sets, or we can do the opposite.

The definitions of the Convex-Hull and Convex sets we choose to work with are the following:

- **Convex-Hull:** $CH(X)$ is the Convex-Hull of a set X iff it is the intersection of all the half-planes that contain X.

- **Convex set:** A set X is Convex iff it is identical to its Convex-Hull, i.e., $X = CH(X)$.

The generalization we propose is obtained by replacing the *half-plane* used in the above definition of the Convex-Hull by a generic set $(B^s)^c$, which is the transposed of the complement of any structuring-element B. We denote the generalized Convex sets as B-Convex sets and the generalized Convex-Hull as B-Convex-Hull because of the dependence on the structuring-element B:

- **B-Convex-Hull:** $CH^B(X)$ is the B-Convex-Hull of X iff it is the intersection of all the translations of $(B^s)^c$ that contain X.

- **B-Convex set:** A set X is B-Convex iff it is identical to its B-Convex-Hull ,i.e., $X = CH^B(X)$.

Actually, the B-Convex-Hull, as defined above, is not a new operation; it is known in Mathematical Morphology as the Morphological **Closing**. In other words:

$$CH^B(X) = X \bullet B. \quad (1)$$

If we choose B to be a disc, and make its radius go to infinity, then the above Closing converges to the conventional Convex-Hull (as pointed out in [2, p. 100]), meaning that the conventional Convex-Hull is indeed a particular case of the generalized Convex-Hull.

Table 1 shows that some of the basic properties of the Convex-Hull and of Convex sets are naturally extended to the B-Convex-Hull operation and to B-Convex sets.

Known Property	Property of the Proposed Generalization
$CH(\cdot)$ is idempotent.	$CH^B(\cdot)$ is idempotent.
$CH(X)$ is the "smallest" convex set that contains X.	$CH^B(X)$ is the "smallest" B-Convex set that contains X.
X is convex iff any two points x and y belonging to X are connected by a segment contained in X. in other words: X is convex iff $\forall\{x,y\} \subseteq X$, $CH(\{x,y\}) \subseteq X$.	If X is B-Convex, then $\forall\{x,y\} \subseteq X$, $CH^B(\{x,y\}) \subseteq X$.
The intersection of convex sets is a convex set.	The intersection of B-Convex sets is a B-Convex set.
X is convex iff every point outside X can be separated from X by a half-plane, i.e., $x \notin X \Rightarrow \exists$ a half-plane that contains x and does not intersect X.	X is B-Convex iff every point outside X can be separated from X by a translation of B^s, i.e., $x \notin X \Rightarrow \exists z \in \mathcal{R}^2$ such that $(B^s)_z$ contains x and does not intersect X.

Table 1: Properties of Convex-Hull and Convex sets.

3. Extreme Points

3.1 Definition and Calculation

Like the Convex-Hull and Convex Sets, there are many ways to define *Extreme Points* of a Convex Set. Table 2 shows one of the classical definitions of Extreme Points for conventional Convex sets, and presents its generalization for B-Convex sets. We denote the set of Extreme Points of a given Convex Set Y by $\mathcal{E}(Y)$ and the set of Extreme Points of a given B-Convex Set X by $\mathcal{E}^B(X)$.

Extreme Points	
Convex sets	B-Convex sets
A point t is an Extreme Point of a Convex set X iff the set $(X - \{t\})$ is also convex.	A point t is an Extreme Point of a B-Convex set X iff the set $(X - \{t\})$ is also B-Convex.

Table 2: Extreme Points

The following *Morphological closed-form formulæ* provide two ways of calculating the set of Extreme Points of a given B-Convex Set X:

$$\mathcal{E}^B(X) = X - \left[\bigcap_{x\in X} (X - \{x\}) \oplus B\right] \ominus B^s \quad (2)$$

$$\mathcal{E}^B(X) = X - \left[\bigcap_{b\in B} X \oplus (B - \{b\})\right] \ominus B^s \quad (3)$$

The outline of the proofs of (2) and (3) are given in appendices A and B, respectively.

If we consider the computational efficiency of the above equations, when implemented on a computer, then (2) is preferable over (3) if X contains fewer elements than B, and (3) is preferable over (2) otherwise.

3.2 Reconstruction from Extreme Points

If a conventional Convex set Y is bounded, then it can be reconstructed back from its Extreme Points by performing the Convex-Hull operation, i.e., $CH(\mathcal{E}(Y)) = Y$. The set of Extreme Points can be seen as a compact representation of a Convex set.

For a B-Convex Set X, a *necessary* condition for perfect reconstruction from its set of Extreme Points $\mathcal{E}^B(X)$ is: $[X - \mathcal{E}^B(X)] \ominus B = \emptyset$. This suggests that X should be "smaller" (in a certain way) than B. Notice that the erosion of any bounded shape by a half-plane is always empty.

The above considerations motivate the definition of a *Reconstruction Window* for a given structuring-element B, inside which every B-Convex Set can be reconstructed from its Extreme Points. A B-Convex Set W is called a Reconstruction Window for B iff $\forall X$ B-Convex, $CH^B[\mathcal{E}^B(X \cap W)] = X \cap W$.

For example, if B is a rectangle, then B itself is a Reconstruction Window for B. If B is a *discrete* rectangle of integer sides n and m, then any discrete rectangle of sides i and j, such that $0 \le i \le (n+1)$ and $0 \le j \le (m+1)$, is a Reconstruction Window for B.

4. Application: Skeleton Redundancy Reduction

It is well-known that the Skeleton representation of images usually contains redundant points, i.e., some of its points may be discarded and still a perfect reconstruction can be obtained [3].

In [3], Maragos and Schafer introduced the concept of *Minimal Skeleton*, which is defined as any subset of the Skeleton *containing no redundant points*, from which perfect reconstruction of the original image is possible. They also presented an efficient algorithm for obtaining a Minimal Skeleton, from a given Skeleton. However, this algorithm is not fully morphological, and therefore can not be implemented on a parallel morphological machine. *Fully morphological* methods for reducing the Skeleton redundancy are studied in [4] and [5].

In the sequel we present an algorithm for *morphologically* obtaining a redundancy-reduced skeleton, based on the B-Convexity theory discussed above.

4.1 The Algorithm

The algorithm is presented below, together with an example. Figure 1 shows the steps of the algorithm for the example.

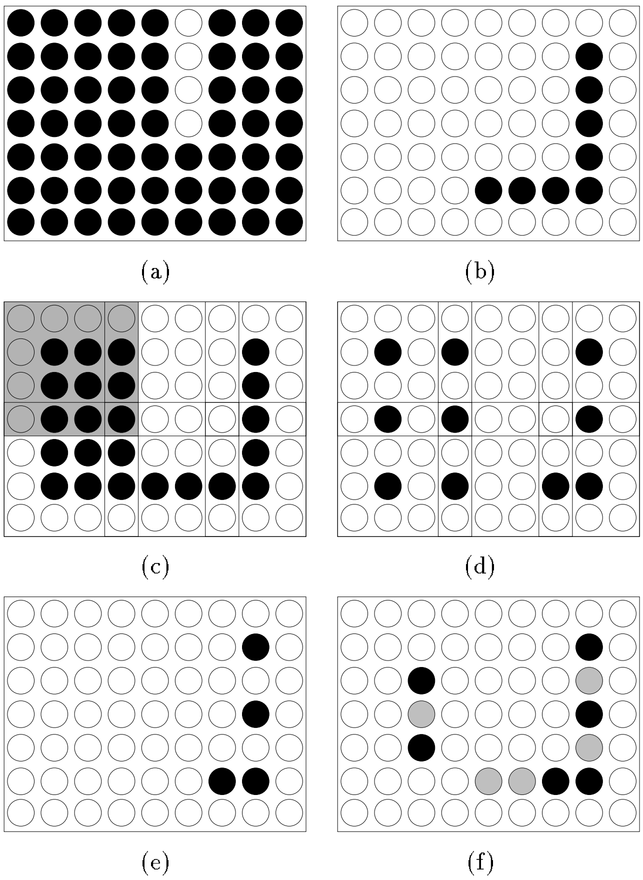

Fig. 1: Proposed algorithm. (a) A discrete binary shape (black dots: foreground, white dots: background), (b) S_1, (c) Z_1, and the partition blocks, (d) Extreme Points of the blocks, (e) $\tilde{S}_1$, and (f) resulting reduced skeleton (black points) compared to the original skeleton (black and grey points).

1. Let X be a given binary image. Choose a structuring-element B, and a family of Reconstruction Windows $\{W^{(nB)}\}$ for all the dilations nB of B. (In the example, X is the digital binary shape shown in Fig. 1(a) (described by the black dots), B is a 3×3 square, and $W^{(nB)}$ are $(2n+2) \times (2n+2)$ squares). Set $n = 0$.

2. Calculate the skeleton subset $S_n \stackrel{\triangle}{=} X \ominus nB - (X \ominus nB) \circ B$, and the set $Z_n \stackrel{\triangle}{=} X \ominus nB$. If Z_n is empty then stop. (In the example, for $n = 1$, S_1 is shown in Fig. 1(b) and Z_1 is seen in Fig. 1(c)).

3. Obtain a partition of Z_n into blocks Y_p^n such that: Y_p^n is the contents of Z_n inside the Reconstruction Window $W^{(nB)}$ centered at p, i.e., $Y_p^n = [W^{(nB)}]_p \cap Z_n$, and the blocks cover the whole set Z_n, i.e., $\bigcup_p Y_p^n = Z_n$. (In the example, the blocks Y_p^n were obtained by translating the Reconstruction Window horizontally and vertically by steps of $p = 2n+1$ pixels, so that there is a 1-pixel-wise overlapping between the blocks. The overlapping by one pixel contributes for the redundancy reduction. Fig. 1(c) shows the first block in grey, and the thin solid lines indicate the position of the other blocks.)

Fig. 2: (a) A binary image and its skeleton, using a 3×3 squared structuring-element, (b) a reduced skeleton obtained by the proposed algorithm.

4. Calculate the Extreme Points of every block Y_p^n, according to nB, $\mathcal{E}^{nB}(Y_p^n)$. Note that Y_p^n is a (nB)-Convex set, since it is the intersection of two (nB)-Convex sets. (Fig. 1(d) shows the result of this operation in the example).

5. Define $C_n \stackrel{\triangle}{=} \bigcup_p \mathcal{E}^{nB}(Y_p^n)$ to be the set of the resulting Extreme Points of all the blocks, and intersect it with the skeleton subset S_n, obtaining $\tilde{S}_n = C_n \cap S_n$. (Fig. 1(e) shows $\tilde{S}_1$).

6. Increment n, and go to 2.

The collection of sets $\{\tilde{S}_n\}$ is the Redundancy-Reduced Skeleton. For comparison between $\{\tilde{S}_n\}$ and the original skeleton $\{S_n\}$, in the scope of the above example, Fig. 1(f) shows the reduced skeleton composed of black dots, and the original skeleton, composed by both the black and grey dots. The grey dots are redundant points removed by the above algorithm.

Exactly as for the conventional Skeleton, the following relation holds:

$$\bigcup_{n \geq k} \tilde{S}_n \oplus nB = X \circ kB \qquad (4)$$

which guarantees partial ($k > 0$) and perfect ($k = 0$) reconstruction of the original image.

4.2 Simulation

Figure 2(a) shows a binary image (Most-significant bit-plane of 256×256-pixel "House"), and its morphological skeleton, calculated with a 3×3 squared structuring-element. The skeleton contains 3173 points.

Fig. 2(b) shows the result of applying the above algorithm to the same binary image. The structuring-element and the Reconstruction Windows are the same as in the example of Fig. 1. The resulting skeleton fully represents the original binary image, and contains 1533 points, i.e., only 48% of the points in the original skeleton.

For comparison, a Minimal Skeleton of the above image, using the non-morphological algorithm given in [3], was calculated. It contains 1362 points, i.e., 43% of the points in the original skeleton, and 89% of the number of points in the proposed reduced skeleton. According the above numbers, the proposed skeleton was able to remove 91% of the redundant points in the original skeleton.

5. Conclusion

A generalization of Convexity is presented, where some of the properties of Convex Sets are extended to sets which are not convex in the traditional sense. Extreme Points of the generalized convex sets are defined, and their ability to fully represent the original set is considered.

Furthermore, an algorithm, based on the above notions, is proposed for *morphologically* reducing the amount of redundant points in the skeleton. Simulation results indicate that most of the redundancy in the skeleton is removed by the proposed algorithm, which is fully morphological.

The proposed approach is also suitable for *morphological* calculation of the set of *Essential Points* of the Skeleton, which is the set of points none of which can be removed from the original skeleton if a perfect reconstruction is desired [6].

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Appendix A

Proof of equation (2)

The set $(X - \{x\}) \bullet B$, for $x \in X$, X a B-Convex set, can be equal either to $(X - \{x\})$ or to X. This is because:

$$X - \{x\} \subseteq (X - \{x\}) \bullet B \subseteq \subseteq X \bullet B = X \quad \text{(A.1)}$$

By definition of Extreme Points, $(X - \{x\}) \bullet B$ is equal to $(X - \{x\})$ iff x is an Extreme Point. Otherwise it is equal to X. Therefore:

$$X - \left[\bigcap_{x \in X} (X - \{x\}) \oplus B\right] \ominus B^s = X - \left[\bigcap_{x \in \mathcal{E}^B(X)} (X - \{x\})\right] = \bigcup_{x \in \mathcal{E}^B(X)} \{x\} = \mathcal{E}^B(X) \quad \text{(A.2)}$$

Appendix B

Outline of the proof of equation (3)

It is enough to prove that for any sets A and B:

$$\bigcap_{a \in A} (A - \{a\}) \oplus B = \bigcap_{b \in B} A \oplus (B - \{b\}) \quad \text{(B.1)}$$

First, let us denote the left hand of the above equation as $\mathcal{H}$, and then write the dilation explicitly in the following way:

$$\mathcal{H} = \bigcap_{a \in A} \bigcup_{b \in B} \bigcup_{\tilde{a} \in A, \tilde{a} \neq a} \{\tilde{a} + b\} \quad \text{(B.2)}$$

Then, after some logical and set manipulations, we notice that a point $z = \tilde{a} + b$ belongs to $\mathcal{H}$ iff there is another pair of points a and $\tilde{b}$, in A and B respectively, such that $a + \tilde{b} = z$. In other words:

$$\mathcal{H} = \{z = \tilde{a} + b = a + \tilde{b} \in A \oplus B \mid \ a \neq \tilde{a}, \tilde{b} \neq b\} \quad \text{(B.3)}$$

Since equation (B.3) is symmetric, i.e., the roles of $A, a, \tilde{a}$ and $B, \tilde{b}, b$ are respectively interchangeable, then we can interchange the above sets and elements also in the original expression, which provides (B.1).