MULTI-PARAMETER SKELETON DECOMPOSITION

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Abstract. In this work we propose a further generalization of the *Morphological Skeleton Decomposition* on Boolean Lattices. In this generalization, the family of structuring-functions which determines the decomposition has its scalar index λ replaced by a *generic* index i from a totally *or partially* ordered set I. This enables, for example, skeleton decompositions by richer and more complex families of shapes, characterized by several scalar indices (i.e., a multi-dimensional index), instead of just a single scalar index.

Particular cases and applications are discussed.

Key words: Skeleton, Mathematical Morphology, Boolean Lattices, Set Decomposition, Image Representation

1. Introduction

The study of the Morphological Skeleton was divided historically into two main branches: topological and algebraic [2]. While characteristics like skeleton connectivity and shape concern the topological branch, the main interest of the algebraic branch in the Skeleton is in it being a compact error-free representation of the original set. In this work, we relate basically to the latter.

According to Serra [2], in order to obtain an appropriate representation, from the above algebraic point of view, the following requirements concerning a Skeleton Decomposition are expected to be satisfied:

1. *Existence and uniqueness* of the Skeleton of a set, for a given family of structuring-functions.
2. *Perfect reconstruction* of the original set from the Skeleton subsets.
3. An *explicit formula* for computing the Skeleton.

These requirements were shown by Lantuéjoul (see [5, Chapter 11]) to be fulfilled when the family of elements are open balls in the Euclidean space, and the sets to be decomposed are open sets in the same space. With the generalization of Mathematical Morphology to Complete Lattices, sufficient conditions for the fulfillment of the requirements were stated in [1, Chapter 2]. Yet, with the advent of new developments in the Skeleton theory [6, 8], Serra recently updated those conditions in [2], providing a quite general framework for the Skeleton Decomposition on Boolean Lattices.

During the above historical development, the family of decomposition elements was always indexed by a non-negative *scalar* parameter λ. The fact that no attempt was made previously to replace this scalar parameter by a vector one can be explained, perhaps, by the interest in obtaining a meaningful Skeleton from the *topological* and *geometrical* points of view (having, for instance, a related *quench function*, which maps every skeleton point to the radius of the respective maximal element). However, the Multi-Structuring-Element Skeleton (MSES) proposed by the authors in [3] (see also [7]) *cannot* be considered as a particular case in Serra's framework (it demands a partial ordering of the indices of the decomposition subsets), and, nevertheless, it satisfies the three *algebraic* requirements stated above.

Our purpose in this paper is to propose a generalized definition of the Skeleton, having as particular cases both Serra's general Skeleton, and the MSES. Therefore, we generalize the above parameter λ (and, therefore, extend the Skeleton scope), while adapting the conditions stated in [2], to assure that the above three requirements are still satisfied. This provides a more flexible analysis tool, and new decompositions besides the above mentioned ones - Multi-Parameter Skeletons and Hybrid Skeletons, as described in the sequel.

The proposed generalization of the Skeleton Decomposition is presented in section 2.1. In section 2.2, theorems assuring that the algebraic requirements are met are presented. In section 3, particular cases of the general decomposition are presented, and in section 4, some applications are discussed.

2. Proposed Generalization

2.1. Generalized Skeleton Definition

Let us consider a set E and the Boolean Lattice defined by $\mathcal{P}(E)$ (the set of subsets of E), *inclusion* order, and *union* and *intersection* as the supremum and the infimum, respectively.

Let I be a set of indices i, totally *or partially* ordered by an order $\geq$. The set I can be, for instance, a d-dimensional space, so that the indices i are vectors.

We choose an arbitrary family of structuring-functions (i.e. functions from E into $\mathcal{P}(E)$), indexed by I, $\{\delta_i(x)\}_{i\in I}$. This choice uniquely determines the families of dilations $\{\delta_i(X)\}$, adjoint erosions $\{\varepsilon_i(X)\}$ and morphological openings $\{\gamma_i(X)\}$, where $X \in \mathcal{P}(E)$.

We define an *element* with "radius" $i \in I$ and "center" $x_0 \in E$, $\delta_i(x_0)$, as the image of x_0 by the structuring-function $\delta_i(x)$, and a *maximal element* in a set $X \in \mathcal{P}(E)$ as an element contained in X but not contained in any other element which is contained in X.

As usual, we define the Skeleton of a set $X \in \mathcal{P}(E)$ as the collection of the centers of all the maximal elements contained in it. It always exists and it is unique for the chosen family of structuring-functions and the given set X. The sets $\{S_i(X)\}_{i\in I}$, each containing the centers of the maximal elements of "radius" i, are called *Skeleton Subsets*.

The above definitions (and also the theorems in next section, and their proofs) are a direct generalization of those given in [2]. The main difference is that, in [2], the set of indices I are restricted to $\mathcal{R}_+$ or $\mathcal{Z}_+$ and its order $\geq$ to the usual total order, whereas here I and $\geq$ are not restricted. Because of lack of space, we do not present first the main results of [2], but proceed with our generalization. However, since the work here follows to a great extent the work in [2], one can easily recover the earlier results by setting $I = \mathcal{R}_+$ or $I = \mathcal{Z}_+$ throughout this paper.

2.2. Calculation and Reconstruction Formulæ

The following theorem provides an explicit formula for calculating the skeleton subsets $\{S_i\}_{i\in I}$:

Theorem 1 *If the family of structuring functions $\{\delta_i(x)\}_{i\in I}$ satisfies the following two conditions:*

1. $i \leq j \Rightarrow \gamma_i\delta_j(x) = \delta_j(x),\ \forall x \in E$
   (i.e., the family is granulometry-generating *in a generalized sense),*
2. *For all $i, j \in I$ and $x, y \in E$:*

$$\delta_i(x) \supseteq \delta_j(y) \Rightarrow \begin{cases} i > j, & \text{if } x \neq y \\ i \geq j, & x = y \end{cases}$$

then, for any $X \subset \mathcal{P}(E)$, and for all $i \in I$:

$$S_i(X) = \varepsilon_i(X) - \bigcup_{j>i\in I} \gamma_{[i,j]}\varepsilon_i(X) \tag{1}$$

where $\gamma_{[i,j]}$ is the morphological opening associated with the structuring function $\delta_{[i,j]}(x) \triangleq \varepsilon_i\delta_j(x)$.

Equation (1) is a direct adaptation of the generalized Lantuéjoul's formula in [2]. The proof is given in the appendix.

A similar direct adaptation of the conditions for lossless reconstruction proposed in [2] guarantees perfect reconstruction from the proposed generalized Skeleton Subsets. In order to state this, by means of a theorem, we consider the following definitions:

- Let L_X be the set of all the elements contained in X, i.e., $L_X \triangleq \{\delta_i(x) \mid i \in I, x \in \varepsilon_i(X)\}$.
- A subset $\{\delta_{i_k}(x_k)\}_{k\in K}$ of L_X is called an *increasing chain* in L_X, if K is a totally ordered set of indices, and if $k < k'$ implies $\delta_{i_k}(x_k) \subseteq \delta_{i_{k'}}(x_{k'})$.
- We say that an increasing chain $\{\delta_{i_k}(x_k)\}$ *converges* to an element $\delta_{i_0}(x_0)$, if $\bigcup_{k\in K} \delta_{i_k}(x_k) = \delta_{i_0}(x_0)$.
- L_X is said to be *inductive for inclusion* [2], if *every* increasing chain in L_X converges to a unique element in L_X.
- Let J be any subset of the set of indices I. We say that J is an *anti-umbra* in I if the conditions $j \in J$ and $i \geq j$ imply $i \in J$.
- For an anti-umbra J in I, we define the following opening: $\forall X \in \mathcal{P}(E)$, $\gamma_J(X) \triangleq \bigcup_{j\in J} \gamma_j(X)$

Theorem 2 *Let $\{S_i(X)\}_{i\in I}$ be the skeleton decomposition of X, according to the family of structuring functions $\{\delta_i(x)\}_{i\in I}$. If L_X is inductive for inclusion, and J is an anti-umbra in I, then*

$$\bigcup_{i\in J} \delta_i(S_i(X)) = \gamma_J(X) \tag{2}$$

The perfect reconstruction is assured whenever $X = \gamma_J(X)$. Equation (2) also provides a *partial reconstruction* formula for the skeleton, in case that $\gamma_J(X) \neq X$. The proof of theorem 2 is also given in the appendix.

3. Multi-Parameter Skeleton

As stressed before, choosing I to be a totally ordered set of indices, such as $\mathcal{R}_+$ or $\mathcal{Z}_+$, brings us to the standard skeleton decomposition, as defined in [2]. The purpose of this paper is to propose decompositions which are *not* based on totally ordered sets.

We consider in this section the following particular case:

- I is a set of d-dimensional vectors, i.e., every index i is in the form $i \triangleq (i_1, i_2, \ldots, i_d)$. More specifically, $I = \mathcal{R}_+^d$ in the continuous case, and $I = \mathcal{Z}_+^d$ in the discrete case.
- The order of I is chosen to be its *strong order*, i.e., for any two indices $i^{(1)}$ and $i^{(2)}$ in I, $i^{(1)} \geq i^{(2)}$ iff $i_n^{(1)} \geq i_n^{(2)}$ for all $n = 1, 2, \ldots, d$.
- The conditions of theorems 1 and 2 hold.

We denote a skeleton associated with such case a *Multi-Parameter Skeleton.* If we denote as k_n the d-dimensional vector having its n-th component equal to 1 and all the others equal to 0, then the *continuous-case* Multi-Parameter Skeleton subsets are given by:

$$S_i(X) = \varepsilon_i(X) - \bigcup_{n=1}^{d} \left[\bigcup_{\lambda_n > 0} \gamma_{[i, i+\lambda_n k_n]} \varepsilon_i(X) \right], \qquad i \in \mathcal{R}_+^d \tag{3}$$

where λ_n, $n = 1, \ldots, d$, are *scalar* variables. Notice that the union in (1), which is performed over the *vector* variable j, is replaced in (3) by unions performed over *scalar* variables, n and λ_n, $n = 1, \ldots, d$.

The unions performed over λ_n in (3) express, actually, convergence towards a supremum, when $\lambda_n \to 0$. In the *discrete case*, these unions converge to maxima, which are reached when $\lambda_n = 1$, $n = 1, \ldots, d$. Therefore, in the discrete case, (3) becomes:

$$S_i(X) = \varepsilon_i(X) - \bigcup_{n=1}^{d} \gamma_{[i, i+k_n]} \varepsilon_i(X), \qquad i \in \mathcal{Z}_+^d \tag{4}$$

We consider now a more specific particular case. Let us select d convex structuring-elements B_n, $n = 1, \ldots, d$, and define for each i in I:

$$A_i \triangleq i_1 B_1 \oplus \ldots \oplus i_d B_d, \tag{5}$$

where $\oplus$ denotes Minkowski addition, and $i_n B_n$, $n = 1, \ldots, d$, denote $(i_n - 1)$ times the dilation of the structuring-element B_n by itself. If the structuring-functions used for the skeletonization are in the form:

$$\delta_i(x) = A(i) \oplus \{x\}, \; i \in I \tag{6}$$

then the associated Multi-Parameter Skeleton is called *Multi-Structuring-Element Skeleton* (MSES) [3, 7]. In the *discrete case* ($I = \mathcal{Z}_+^d$), the morphological openings $\gamma_{[i,i+k_n]}(X)$ associated with a MSES are independent of i, and equal to $X \circ B_n$ (where $(\cdot) \circ B$ denotes translation invariant morphological opening by the structuring-element B). Therefore, if we denote by $(\cdot) \ominus B$ the translation invariant erosion by

B, the associated MSES subsets $S_i(X)$ are given by:

$$S_i(X) = X \ominus A_i - \bigcup_{n=1}^{d} [X \ominus A_i] \circ B_n, \qquad i \in \mathcal{Z}_+^d. \tag{7}$$

3.1. Example

As an example, let us consider a MSES with $d = 2$, B_1 equal to the unit *vertical* open segment, and B_2 equal to the unit *horizontal* open segment.

The related Skeleton decomposition of an open set X yields the centers of the maximal open *rectangles* inscribable in X (note that $(i_1 B_1 \oplus i_2 B_2)$ in this case is a rectangle with vertical side i_1 and horizontal side i_2).

In the discrete case ($I = \mathcal{Z}^2$), the decomposition is with respect to rectangles with sides of discrete lengths only, and it can be calculated using (7).

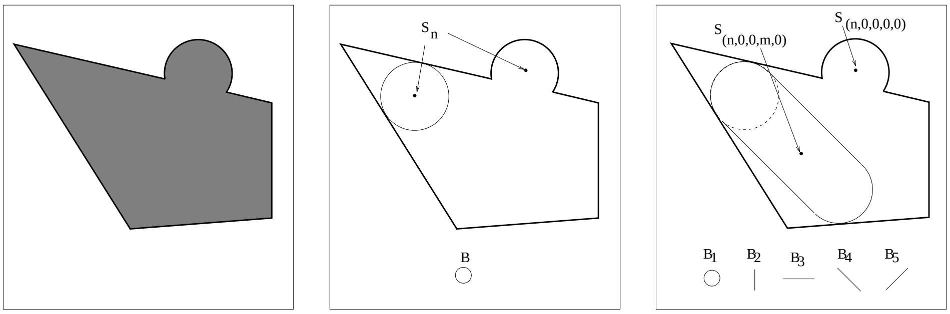
A question which can be asked at this point is: Would we get the same decomposition as above if, instead of using (7), we calculated the Skeleton w.r.t. the family $\{\lambda B_1\}_{\lambda \in \mathcal{R}_+}$ of vertical open segments, and then, *on the resultant subsets*, the Skeleton w.r.t. the family $\{\lambda B_2\}_{\lambda \in \mathcal{R}_+}$ of horizontal open segments? The answer is no; composition of skeletons do not give centers of *maximal elements.* For a decomposition into maximal elements, the composite skeleton must be calculated.

4. Applications

The applications of the conventional skeleton (I being a totally ordered 1-dimensional space), are well known. In this section, we present some applications of skeletons based on partially ordered sets, such as Multi-Parameter Skeletons.

From the *topological* and *geometrical* points of view, a Multi-Parameter Skeleton may have little interest. First, it is far less connectivity-preserving than the conventional skeleton. Also, it fails to exactly provide a *Medial Axis* of the shapes under study. And, finally, a Multi-Parameter Skeleton usually can *not* be characterized by a *quench function*, because a point $x \in E$ may be the "center" of two (or more) different maximal elements. The latter is consequence of I being *partially* ordered.

On the other hand, from the *algebraic* point of view, a Multi-Parameter Skeleton can be very useful. First, it can decompose an image into an assortment of shapes richer than the one a conventional skeleton is able to provide. Moreover, if we consider the element indices i to be a degree of "importance", or as a "category classifier", as is often done regarding the conventional skeleton, then a Multi-Parameter Skeleton can provide finer distributions and classifications. In addition, there is greater diversity of possible partial reconstructions, where their proximity to the original image are controlled by the choice of the anti-umbra J in (2). Finally, each of the scalar parameters of the multi-dimensional index i may have a different physical interpretation, such as size, time duration, gray-level, etc., in contrast to the conventional skeleton decomposition, where different physical characteristics of the image can not be treated independently.



(a) (b) (c)

Fig. 1. (a) A binary image containing a partially occluded disc, (b) elements detected by a conventional skeleton, (c) elements detected by a 5-parameter skeleton.

4.1. Pattern Recognition

Suppose we are interested in finding a pattern in a binary image, and that this pattern is not corrupted with holes but may be partially occluded. E.g., the disc in Figure 1(a).

In order to detect the shape, let us consider a family of translation invariant structuring-functions, with the structure shown in (6). If the pattern we are interested in is one of the shapes of the family $\{A_i\}$, then it should be easy to locate such a pattern using the associated skeleton decomposition, since it provides the centers of maximal elements from that family in the given image. In other words, we consider patterns to be located as maximal elements, and define a proper family of shapes $\{A_i\}$ for decomposition. Thus, in order to detect the disc in Fig. 1, we may calculate the skeleton w.r.t. a family of increasing discs.

However, for a conventional skeleton, the above idea does not work well. In Fig. 1(b) we see that we may find, in the subsets S_n of the skeleton, maximal elements other than the pattern we are looking for.

On the other hand, a MSES could give better results. For the example in Fig. 1, we choose a 5-parameter skeleton, based on 5 structuring-elements: a unit disc (which we want to detect), and 4 unit lines in four different directions. In this case, we are interested in the subsets of the form $S_{(n,0,0,0,0)}$ only. As seen in Fig. 1(c), most of the "false alarms" obtained by the conventional skeleton are now in different subsets than $S_{(n,0,0,0,0)}$.

4.2. Coding

Let us compare a Multi-Parameter Skeleton, w.r.t. a partially ordered family of structuring-functions $\mathcal{F}$, to a conventional one-parameter skeleton w.r.t. a totally ordered sub-family $\tilde{\mathcal{F}}$ *contained* in $\mathcal{F}$. For example, if we consider the family of open rectangles defined in section 3.1 to be $\mathcal{F}$, then $\tilde{\mathcal{F}}$ could be the family of open squares. Because of the partial ordering, the number of skeleton points in the multi-parameter skeleton is expected to be larger than the number of skeleton points in the one-parameter skeleton. But after removing redundant points in both skeletons (see [4, 7] for redundancy reduction approaches), the situation is inverted; the multi-parameter skeleton is expected to contain considerably fewer points than the one-parameter skeleton, which appears to be of great advantage for Coding purposes.

However, since the number of subsets in the multi-parameter skeleton is usually much bigger than the number of subsets in the one-parameter skeleton (about N^d in comparison to N), this turns out to be too costly in terms of coding efficiency. Moreover, its computational burden, usually of order $\mathcal{O}(d)$, is very high if compared to $\mathcal{O}(1)$ of the one-parameter skeleton.

However, the General Skeleton Decomposition presented in section 2.1 does not restrict us to either a one-parameter or a multi-parameter skeleton; combinations of them are also possible. For example, instead of considering a decomposition w.r.t. the family of *all* the rectangles, as presented in section 3.1, or w.r.t. the family of squares only, We can *arbitrarily* select *any* sub-family of rectangles for the skeletonization. This combines, at some extent, the advantages of both the multi-parameter and the one-parameter skeletons. We call such decomposition a Hybrid Skeleton.

As opposed to a full multi-parameter decomposition, which in the general case is not substantially advantageous when compared to the one-parameter skeleton, Hybrid Skeletons showed promising results in preliminary simulations. Methods for optimal (or sub-optimal) determination of the sub-family of structuring-functions (given a full multi-parameter family) are presently being examined.

5. Conclusion

The scope of the Morphological Skeleton definition is further extended, so that the family of structuring-functions which determines the decomposition can be indexed by a *generic* index, instead of a *scalar* one only. Conditions for the morphological calculation of the Generalized Skeleton subsets and for the reconstruction of the original image are determined, and the respective formulæ derived.

Particular cases of the general definition (which are called Multi-Parameter Skeletons, Multi-Structuring-Element Skeletons (MSES's), and Hybrid Skeletons) are considered, and potential applications of such skeletons, with comparisons to the application of conventional skeleton, are also presented.

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Appendix

Proof of Theorem 1:

A maximal element from a family $\{\delta_i(x), i \in I, x \in E\}$ inside a set X is an element $\delta_i(x)$ contained in X, such that, for any $j \neq i$ and $y \in E$, if $\delta_j(y) \subseteq X$, then $\delta_i(x) \not\subseteq \delta_j(y)$. But since condition 2 of the theorem makes it impossible for $\delta_j(y)$ to contain $\delta_i(x)$ if $j \not> i$ (unless $i = j$ and $x = y$), we need to check only for $j > i$. The skeleton subset $S_i(X)$ is the set of points x, such that $\delta_i(x)$ is maximal in X.

The following is always true:

$$\delta_i(x) \subseteq X \Leftrightarrow x \in \varepsilon_i(X) \tag{A.1}$$

$$\delta_i(x) \not\subseteq \delta_j(y) \Leftrightarrow x \notin \varepsilon_i\delta_j(y) = \delta_{[i,j]}(y) \tag{A.2}$$

It is always true also that $\delta_j(y) \subseteq X$ implies $\varepsilon_i\delta_j(y) \subseteq \varepsilon_i(X)$, but the equivalence is usually not assured. However, since the family of structuring-functions is granulometry-generating (condition 1 of the theorem), for $j > i$ the equivalence is obtained, which can be written in the following way:

$$\delta_j(y) \subseteq X \Leftrightarrow \delta_{[i,j]}(y) \subseteq \varepsilon_i(X) \Leftrightarrow y \in \varepsilon_{[i,j]}\varepsilon_i(X) \tag{A.3}$$

Therefore, according to (A.1), (A.2) and (A.3), $x \in S_i(X)$ iff $x \in \varepsilon_i(X)$ and $x \notin \gamma_{[i,j]}\varepsilon_i(X)$, $\forall j > i$. This leads to (1).

Proof of Theorem 2:

The left side of (2) can be written in the following way:

$$\bigcup_{i \in J} \delta_i(S_i(X)) = \bigcup_{i \in J} \bigcup_{x \in S_i(X)} \delta_i(x) \tag{A.4}$$

which means that it is equal to the union of all the maximal elements contained in X, with "radius" in J. Therefore, we need to proof that $x \in \gamma_J(X)$ iff x belongs to some maximal element with "radius" in J.

If $x \in X$ belongs to a maximal element $\delta_j(y)$, $j \in J$, then $x \in \gamma_j(X) \subseteq \gamma_J(X)$, which proves one way.

If $x \in \gamma_J(X)$ then there is $j \in J$ such that $x \in \delta_j(y) \in L_X$ (for some $j \in E$), which is not necessarily a maximal element. But, since L_X is inductive for inclusion (condition of the theorem), every increasing chain in L_X converges to a unique element, and therefore, by Zorn's Lemma, $\delta_j(y)$ (and hence x too) is contained in a maximal element. The radius of this maximal element is obviously greater or equal to j and belongs to I. Thus, since J is an anti-umbra in I (condition of the theorem), the radius of the maximal element containing x belongs to J, which proves the way back.

The above proofs are an extension of those in [2].