Best Basis Algorithm for Orthonormal Shift-Invariant Trigonometric Decomposition

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ABSTRACT

The Local Cosine Decomposition of Coifman and Wickerhauser [1] is modified by incorporating two degrees of freedom that increase the adaptability of the best basis. These are relative shifts between resolution levels and adaptive polarity foldings. The resultant expansion is shift-invariant, and yields adaptive time-frequency distributions which are characterized by high resolution, high concentration and suppressed cross-terms associated with the Wigner distribution.

1. INTRODUCTION

Adaptive signal representations in overcomplete libraries of waveforms have been widely used in recent years. Instead of representing a prescribed signal in a fixed basis, it is useful to choose a suitable basis that facilitates a desired application, such as compression, identification, classification or noise removal (denoising) [1, 2, 3]. Coifman and Meyer were the first to introduce libraries of orthonormal bases which are organized in a binary tree structure, where the best basis can be efficiently searched for, and whose elements are localized in the time-frequency plane [4]. Of particular interest are the libraries of local trigonometric bases, which consist of sines or cosines multiplied by smooth window functions, and libraries of wavelet packet bases, comprising translations and dilations of wavelet packets. The libraries are naturally organized in binary trees, and the best basis which minimizes a certain information cost function (e.g., entropy) is searched using the divide-and-conquer algorithm [1].

A serious drawback is the lack of shift-invariance. Both the wavelet packet decomposition (WPD) and local cosine decomposition (LCD) of Coifman and Wickerhauser [1], as well as the extended algorithms proposed by Herley *et al.* [5], are sensitive to the signal location with respect to the chosen time origin. Several approaches that resolve this problem (*e.g.*, [2, 6, 7]) either introduce redundancy (oversampling), entail high computational complexity, or alternatively the resulting representations are non-unique and involve approximate signal reconstructions.

Another approach is to extend the library of bases, in which the best representations are searched for, by introducing an additional degree of freedom that adjusts the time-localization of the basis functions [8, 9, 10, 11]. In case of wavelet transform or wavelet packet decomposition, the additional degree of freedom is possibly an adaptive even-odd down-sampling. That is, following the low-pass and high-pass filtering, when expanding a parent-node, retain either all the odd samples or all the even samples, according to the choice which minimizes the cost function. This modification of the wavelet transform and wavelet packet decomposition leads to orthonormal best-basis representations which are shift-invariant and characterized by lower information costs [9].

In this work a similar approach is applied to smooth local trigonometric bases. We modify the LCD of Coifman and Wickerhauser by incorporating into the best-basis search algorithm two additional degrees of freedom that increase the adaptability of the best-basis. These degrees of freedom are relative shifts between expansions in distinct resolution levels and adaptive polarity foldings. It is proved that the proposed algorithm, namely *shift-invariant adapted-polarity local trigonometric decomposition* (SIAP-LTD), leads to a "best-basis" representation that is shift-invariant, orthonormal and characterized by a lower information cost and improved time-frequency resolution when compared to the LCD.
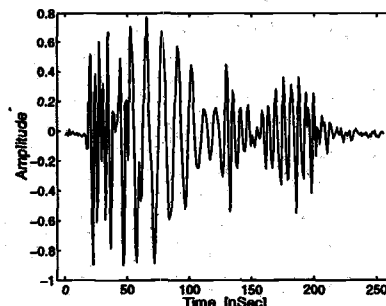

Figure 1. The signal $g(t)$: Evolution of an electromagnetic pulse in a relativistic magnetron (heterodyne detection; local oscillator = 2.6GHz).
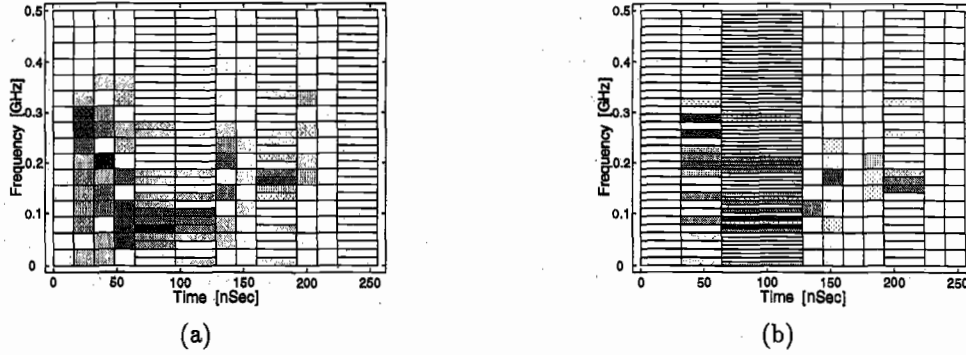
Figure 2. Effect of a temporal shift on the time-frequency representation using the Local Cosine Decomposition: (a) $g(t)$ in its best basis, Entropy=3.30. (b) $g(t - 20)$ in its best basis, Entropy=3.08.

2. THE BEST-BASIS EXPANSION

Let $\mathcal{M}$ denote an additive information cost functional and let $\mathcal{B}$ represent a library of orthonormal bases. Then for a signal g, the best basis in $\mathcal{B}$ is that $B \in \mathcal{B}$ for which $\mathcal{M}(Bg)$, the information cost of representing g on the basis B, is minimal.

For the LCD [1], a prescribed signal supported on an interval $[0, N)$ is first split into overlapping intervals $[2^{J-\ell}n - \epsilon, 2^{J-\ell}(n+1) + \epsilon)$, where ℓ $(0 \leq \ell \leq L \leq J)$ and n $(0 \leq n < 2^\ell)$ are respectively the resolution-level and position indices, $J \equiv \log_2 N$, and $\epsilon > 0$ controls the smoothness of the windows. Then a folding operator [12] "folds" overlapping parts into the segments, and a standard cosine transform is applied on each segment. In this case, the basis-functions are local cosines with even parity at the left endpoint and odd parity at the right endpoint.

Here, the LCD is modified and extended by allowing a larger set of intervals and a variable-polarity folding operator, which is adapted to the signal [10]. The intervals, having an additional degree of freedom are of the form:

$$I_{\ell,n,m} = [2^{J-\ell}n + m - \epsilon, 2^{J-\ell}(n+1) + m + \epsilon) \quad (1)$$

where m $(0 \leq m < 2^{J-\ell})$ is a shift index. In this case, the basis-functions are local sines or cosines with either odd or even parities at the segments' endpoints. To ensure orthonormality of the basis-functions, at each endpoint the parity of basis-functions on the left is opposite to the parity of those on the right. Therefore intervals have at their endpoints either even-odd polarity or odd-even polarity. We designate the polarity at an endpoint by an index $\rho \in \{0, 1\}$ ($\rho = 0$ for even-odd polarity and $\rho = 1$ for even-odd polarity).

Denote by $B_{\ell,n,m}^{\rho_0,\rho_1}$ the set of basis functions on the interval $I_{\ell,n,m}$ having ρ_0 polarity at the left endpoint and ρ_1 polarity at the right endpoint. Let $A_{\ell,n,m}^{\rho_0,\rho_1}$ be the best basis for g restricted to the closure of the linear span of $B_{\ell,n,m}^{\rho_0,\rho_1}$. Accordingly,

$A_{0,0,\tau}^{p,p}$ for some shift $0 \leq \tau < N$ and polarity $p \in \{0, 1\}$ constitutes the best basis for g. These parameters, namely $\tau = m_0$ and $p = p_0(0)$ which specify the shift and polarity at the coarsest resolution level ($\ell = 0$), are determined recursively together with the best basis: Suppose that at the resolution level ℓ we have found m_ℓ, $\{p_\ell(i) \mid 0 \leq i \leq 2^\ell\}$ and $A_{\ell,n,m_\ell}^{p_\ell(n),p_\ell(n+1)}$ for all $0 \leq n < 2^\ell$. Then we will choose $m_{\ell-1}$, $\{p_{\ell-1}(i) \mid 0 \leq i \leq 2^{\ell-1}\}$ and $A_{\ell-1,n,m_{\ell-1}}^{p_{\ell-1}(n),p_{\ell-1}(n+1)}$ for $0 \leq n < 2^{\ell-1}$ so as to minimize the information cost.

It can be shown that if I and J are adjacent intervals, then we can switch from a basis on the interval $I \cup J$ to bases on I and J, since $\overline{Span}\{B_I^{\rho_0,\rho_1}\} \oplus \overline{Span}\{B_J^{\rho_1,\rho_2}\} = \overline{Span}\{B_{I \cup J}^{\rho_0,\rho_2}\}$.

By (1) we have that

$$I_{\ell-1,n,m} = I_{\ell,2n+\alpha,m_c} \cup I_{\ell,2n+1+\alpha,m_c} \quad (2)$$

where $m_c = m - \alpha \cdot 2^{J-\ell}$ and $\alpha \in \{0, 1\}$ such that $m_c \in [0, 2^{J-\ell})$. Consequently,

$$A_{\ell-1,n,m}^{\rho_0,\rho_1} = \begin{cases} B_{\ell-1,n,m}^{\rho_0,\rho_1}, & \text{if } \mathcal{M}_B \leq \mathcal{M}_A, \\ A_{\ell,2n+\alpha,m_c}^{\rho_0,\rho_2} \oplus A_{\ell,2n+1+\alpha,m_c}^{\rho_2,\rho_1}, & \text{else} \end{cases} \quad (3)$$

where $\mathcal{M}_A = \mathcal{M}(A_{\ell,2n+\alpha,m_c}^{\rho_0,\rho_2}g) + \mathcal{M}(A_{\ell,2n+1+\alpha,m_c}^{\rho_2,\rho_1}g)$ is the information cost of the children, $\mathcal{M}_B = \mathcal{M}(B_{\ell-1,n,m}^{\rho_0,\rho_1}g)$ the information cost of the parent, and $\rho_2 = p_\ell(2n + 1 + \alpha)$ is the right polarity of the left child and left polarity of the right child. Now, to acquire shift-invariance it is sufficient to consider two optional values of $m_{\ell-1}$: m_ℓ and $m_\ell + 2^{J-\ell}$. The respective information costs of g when expanded at the resolution level $\ell - 1$ are

$$\mathcal{M}'_{\ell-1} = \sum_{n=0}^{2^{\ell-1}-1} \mathcal{M}(A_{\ell-1,n,m_\ell}^{p_\ell(2n),p_\ell(2n+2)}g) \quad (4)$$

$$\mathcal{M}''_{\ell-1} = \sum_{n=0}^{2^{\ell-1}-1} \mathcal{M}(A_{\ell-1,n,m_\ell+2^{J-\ell}}^{p_\ell(2n+1),p_\ell(2n+3)}g) \quad (5)$$
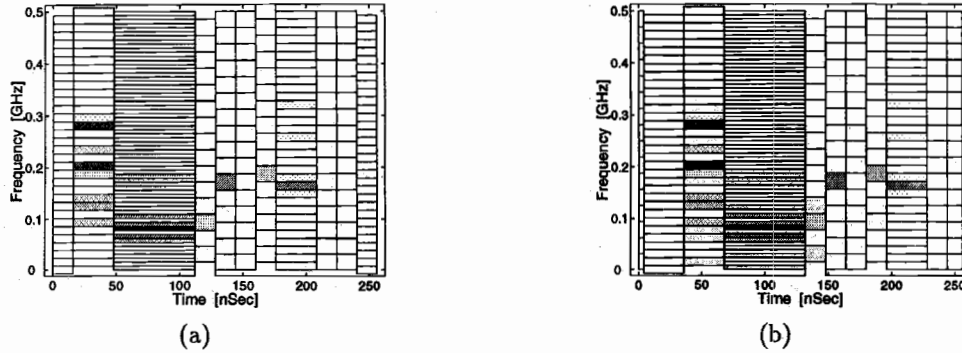
Figure 3. Time-frequency representations using the SIAP-LTD: (a) $g(t)$ in its best basis, Entropy= 2.89. (b) $g(t - 20)$ in its best basis, Entropy= 2.89. Compared with the LCD (Fig. 2), beneficial properties are shift-invariance and lower information cost.

Thus we select that value of $m_{\ell-1}$ which yields a cheaper representation, i.e.,

$$m_{\ell-1} = \begin{cases} m_\ell, & \text{if } \mathcal{M}'_{\ell-1} \leq \mathcal{M}''_{\ell-1}, \\ m_\ell + 2^{J-\ell}, & \text{else.} \end{cases} \quad (6)$$

The polarity indices at the resolution level $\ell-1$ are obtained by keeping those which correspond to endpoints. Namely, for $0 \leq n \leq 2^{\ell-1}$

$$p_{\ell-1}(n) = \begin{cases} p_\ell(2n), & \text{if } \mathcal{M}'_{\ell-1} \leq \mathcal{M}''_{\ell-1}, \\ p_\ell(2n + 1 \bmod 2^\ell), & \text{else.} \end{cases} \quad (7)$$

The recursive procedure is carried out down to a specified level $\ell = L$ $(L \leq J)$, where we impose

$$A^{\rho_0,\rho_1}_{L,n,m} = B^{\rho_0,\rho_1}_{L,n,m} \quad (8)$$

and pick a combination of shift and polarity by

$$m_L = \arg \min_{0 \leq m < 2^{J-L}} \Big\{ \sum_{n=0}^{2^L-1} \mathcal{M}(B^{\pi_m(n),\pi_m(n+1)}_{L,n,m} g) \Big\} \quad (9)$$

$$p_L(n) = \pi_{m_L}(n), \quad 0 \leq n < 2^L \quad (10)$$

where

$$\pi_m(n) \equiv \arg \min_{\pi \in \{0,1\}} C_{m,n}(\pi),$$

$$C_{m,n}(\pi) \equiv \min_{\rho_0,\rho_1 \in \{0,1\}} \{\mathcal{M}(B^{\rho_0,\pi}_{L,n,m} g) + \mathcal{M}(B^{\pi,\rho_1}_{L,n+1,m} g)\}.$$

Proposition 1 *The best basis expansion stemming from the previously described recursive algorithm is shift-invariant.*

Detailed proof appears in [10]. The computational complexity of executing SIAP-LTD is $O(N(L + 2^{J-L+1})\log_2 N)$, where N denotes the length of the signal. This complexity is comparable to that of LCD [1] $(O(NL\log_2 N))$ with the benefits of shift-invariance and a higher quality (lower "information cost") "best-basis".

3. ADAPTIVE TIME-FREQUENCY DISTRIBUTION

We have used the time-frequency representation deriving from the SIAP-LTD to analyze electromagnetic pulses ($\approx$200 nanoseconds long) generated by a high power ($\approx$100 MegaWatts) relativistic magnetron [13]. Time resolved spectral power density sheds more light on undesirable phenomena, such as mode build-up and competition and pulse shortening [14], which are common in these high power short pulse microwave tubes. To illustrate the shift-invariant properties of the SIAP-LTD and its enhanced time-frequency representation compared to the LCD, we refer to the expansions of the signals $g(t)$ (Fig. 1) and $g(t-20)$. For definiteness, we choose entropy as the cost function. Figs. 2 and 3 depict the "best-basis" expansions under the LCD and the SIAP-LTD algorithms, respectively. The sensitivity of LCD to temporal shifts is obvious, while the "best-basis" SIAP-LTD representation is indeed shift-invariant and characterized by a lower entropy.

The tilings of the time-frequency (TF) plane are idealized representations interconnected with specific basis expansions. A basis-function is symbolized by a rectangular cell whose area is associated with Heisenberg's uncertainty principle, and its shade is proportional to the corresponding coefficient squared. To form time-frequency distributions, we sum up the auto Wigner distribution (WD) of the basis functions and cross WD of pairs which are "close" in the TF plane. Since the cross-term interference is caused by the cross WD of distinct components, one can decide on a distance threshold D in the TF plane, such that farther basis-functions are considered unrelated and their cross WD is discarded.

Let $g = \sum_\lambda c_\lambda \varphi_\lambda$ be the best-basis expansion of the signal g. Then its TF distribution is given by

$$\text{TFD}_g = \sum_{\lambda \in \Lambda} |c_\lambda|^2 W_{\varphi_\lambda} + 2 \sum_{(\lambda,\lambda') \in \Gamma} \text{Re}\{c_\lambda c^*_{\lambda'} W_{\varphi_\lambda,\varphi_{\lambda'}}\} \quad (11)$$
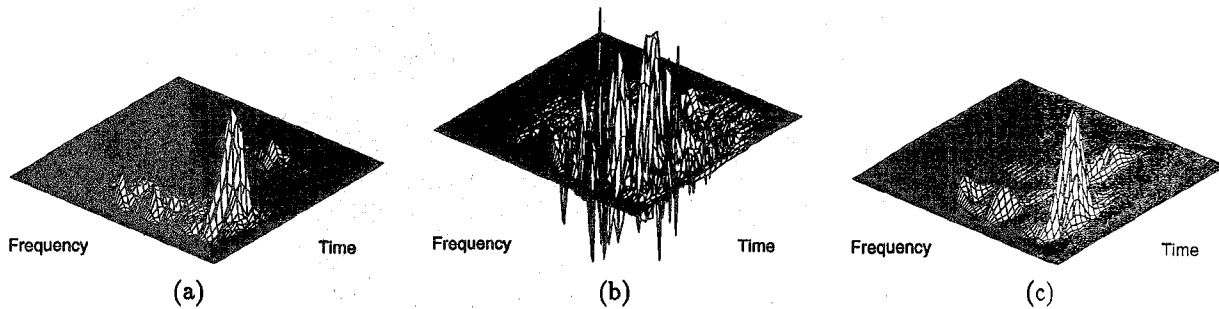
Figure 4. Mesh plots for the signal $g(t)$: (a) SIAP-LTD based time-frequency distribution; (b) Wigner distribution; (c) Smoothed pseudo Wigner distribution. The SIAP-LTD yields an *adaptive* distribution where high resolution, high concentration, and suppressed cross-term interference are attainable.

where the contributing basis-functions are only those whose coefficients are large enough:

$$\Lambda = \{\lambda \mid |c_\lambda| \geq \delta M\}, \quad M \equiv \max_\lambda\{|c_\lambda|\}$$

$$\Gamma = \{(\lambda, \lambda') \mid d(\varphi_\lambda, \varphi_{\lambda'}) \leq D, \ |c_\lambda c_{\lambda'}| \geq \delta^2 M^2\}.$$

By adjusting the distance threshold D and amplitude threshold δ, one can effectively balance the cross-term interference, the useful properties of the distribution (time/frequency marginals, energy conservation, instantaneous frequency, etc. [15]), and the computational complexity. Fig. 4 compares between the SIAP-LTD based time-frequency distribution, the WD and smoothed pseudo Wigner distribution [15] for the signal $g(t)$. The adaptive time-frequency representation presented here obtains high resolution and concentration in time-frequency, and avoids the cross-terms associated with the WD.

4. CONCLUSION

A fast algorithm for shift-invariant best-basis expansion and adaptive time-frequency distribution is presented. When compared with the LCD algorithm proposed in [1], SIAP-LTD is determined to be advantageous in four respects: 1) Shift-invariance. 2) Lower information cost. 3) The folding operator is adapted to the parity properties of the signal. 4) It provides an adaptive time-frequency distribution, attaining high resolution and concentration, and suppressed cross-term interference. The generated distribution is adjusted by threshold parameters that balance the cross-term interference, the useful properties and the computational complexity.

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