

# ADAPTIVE TIME-FREQUENCY DISTRIBUTIONS VIA THE SHIFT-INVARIANT WAVELET PACKET DECOMPOSITION

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## ABSTRACT

Utilizing the *Shift-Invariant Wavelet Packet Decomposition* (SIWPD), various useful properties relevant to time-frequency analysis, including high energy concentration and suppressed interference terms, can be achieved simultaneously in the Wigner domain. A prescribed signal is expanded on its best basis and transformed into the Wigner domain. Subsequently, the interference terms are eliminated by adaptively thresholding the cross Wigner distribution of interactive basis functions, according to their amplitudes and distance in an idealized time-frequency plane. The properties of the resultant *modified Wigner distribution* (MWD) are investigated, and its performance in eliminating interference terms, while still retaining high energy resolution, is compared with that of other existing approaches. We demonstrate the effectiveness of the proposed MWD to resolving multicomponent signals. Each component is determined as a partial sum of basis-functions over a certain equivalence class in the time-frequency plane.

## 1. INTRODUCTION

The Wigner distribution (WD) is of special theoretical importance in time-frequency analysis, because it possesses a number of desirable properties, including maximum auto-component concentration in the time-frequency plane [1]. However, practical applications of the WD are rather limited by the presence of interference terms. The appearance of such terms renders the WD of multicomponent signals extremely difficult to interpret. The reduced-interference distributions [2] were developed to attenuate interference terms by employing some kind of smoothing kernel or windowing. Unfortunately, this reduces the energy concentration of the analyzed signal and dramatically affects the appearance and quality of the resulting representation.

A different approach uses the Gabor expansion to decompose the WD [3]. Interference terms are identified as cross WD of distinct basis functions. A major drawback is the dependence of the performance on the specific choice of the Gabor window. An appropriate window selection depends on the data and may vary for different components of the same signal [3]. Furthermore, the cross-terms of basis functions that are "close" in the time-frequency plane are not always interpretable as interference terms, but rather may have a significant effect on the time-frequency resolution. Qian and Chen [4] proposed to decompose the WD into a series of Gabor expansions, where the order of the expansion is defined by the maximum degree of oscillation. They showed that such harmonic terms contribute minimally to the useful properties, but are directly responsible for the ap-

pearance of interference terms. In this case, the manipulation of cross-terms is equivalent to including cross-terms of Gabor functions whose *Manhattan distance* is smaller than a certain threshold. However, the order of the expansion has to be determined adaptively and generally depends on the local distribution of the signal. In [5], the signal is decomposed into frequency bands, and the Wigner distributions of all the subbands are superimposed. This attenuates interferences between subbands, but still suffers interferences within the subbands. Therefore it would be suitable for signals possessing a single component in each of the subbands. Moreover, the exclusion of beneficial cross-terms, which join neighboring basis-functions, invariably degrades the energy concentration and may artificially split a given signal component into several frequency-bands.

Recently, we have introduced *adaptive* decompositions of the WD using *extended* libraries of orthonormal bases [6]. We showed that the interference terms in the Wigner domain can be reduced by adaptively thresholding the cross WD of pairs of basis functions. The distinction between undesirable interference-terms and beneficial cross terms, which enhance the time-frequency representation, involves an adaptive expansion of the analyzed signal and correspondingly an adaptive distance measure. A prescribed signal is expanded on its best basis and transformed into the Wigner domain. Subsequently, the interference terms are eliminated by thresholding the cross WD of interactive basis functions, according to their amplitudes and distance in the time-frequency plane. The distance measure weighs the Euclidean distance with the time and frequency uncertainties of the basis-functions, and thus dispenses with the need for local adjustments of the associated distance-threshold.

In this paper, the properties of the *modified Wigner distribution* (MWD) are investigated, and its performance in eliminating interference terms, while still retaining high energy resolution, is compared with that of other existing approaches. It is shown that the proposed MWD is directly applicable to resolving multicomponent signals. Each component is determined as a partial sum of basis-functions over an equivalence class in the time-frequency plane.

## 2. MODIFIED WIGNER DISTRIBUTION

Let  $g(t) = \sum_{\lambda} c_{\lambda} \varphi_{\lambda}(t)$  be the *shift-invariant wavelet packet decomposition* (SIWPD) of the signal  $g$  [7]. Then, the MWD for  $g$  is defined by

$$T_g = \sum_{\lambda \in \Lambda} |c_{\lambda}|^2 W_{\varphi_{\lambda}} + 2 \sum_{\{\lambda, \lambda'\} \in \Gamma} \text{Re}\{c_{\lambda} c_{\lambda'}^* W_{\varphi_{\lambda}, \varphi_{\lambda'}}\} \quad (1)$$

where  $W_{\varphi_\lambda}$  is the auto WD of  $\varphi_\lambda$  and  $W_{\varphi_\lambda, \varphi_{\lambda'}}$  is the cross WD of  $\varphi_\lambda$  and  $\varphi_{\lambda'}$ :

$$W_{\varphi_\lambda, \varphi_{\lambda'}}(t, \omega) = \int \varphi_\lambda(t + \tau/2) \varphi_{\lambda'}^*(t - \tau/2) e^{-j\omega\tau} d\tau, \quad (2)$$

$$W_{\varphi_\lambda}(t, \omega) \equiv W_{\varphi_\lambda, \varphi_\lambda}(t, \omega). \quad (3)$$

The summations in (1) are restricted to basis-functions whose coefficients are above a prescribed cutoff, and to pairs which are sufficiently "close" in time-frequency plane. Let  $\epsilon$  and  $D$  denote respectively thresholds of relative amplitude and time-frequency distance. Then the sets  $\Lambda$  and  $\Gamma$  are given by

$$\Lambda = \{\lambda \mid |c_\lambda| \geq \epsilon M\}, \quad M \equiv \max_\lambda \{|c_\lambda|\} \quad (4)$$

$$\Gamma = \{ \{\lambda, \lambda'\} \mid 0 < d(\varphi_\lambda, \varphi_{\lambda'}) \leq D, |c_\lambda c_{\lambda'}| \geq \epsilon^2 M^2 \}. \quad (5)$$

The distance  $d$  between a pair of basis-functions is defined by

$$d(\varphi_\lambda, \varphi_{\lambda'}) = \left[ \frac{(\bar{t}_\lambda - \bar{t}_{\lambda'})^2}{\Delta t_\lambda \Delta t_{\lambda'}} + \frac{(\bar{\omega}_\lambda - \bar{\omega}_{\lambda'})^2}{\Delta \omega_\lambda \Delta \omega_{\lambda'}} \right]^{1/2} \quad (6)$$

where  $(\bar{t}_\lambda, \bar{\omega}_\lambda)$  is the time-frequency position of  $\varphi_\lambda$ ;  $\Delta t_\lambda$  and  $\Delta \omega_\lambda$  denote the corresponding time and frequency uncertainties. Similar notations apply to  $\varphi_{\lambda'}$ .

The SIWPD employs basis-functions of the form

$$\psi_{\ell, n, m, k}(t) = 2^{\ell/2} \psi_n [2^\ell (t - 2^{-L} m) - k] \quad (7)$$

where  $\{\psi_n(t) : n \in \mathbb{Z}_+\}$  are wavelet packets [8],  $\ell$  is the resolution-level index ( $0 \leq \ell \leq L$ ),  $m$  is the shift index ( $0 \leq m < 2^{L-\ell}$ ),  $k$  is the position index ( $0 \leq k < 2^\ell$ ) and  $L$  denotes the finest resolution level. Each basis-function is symbolically associated with a rectangular tile in the time-frequency plane which is positioned about

$$\bar{t} = 2^{-\ell} k + 2^{-L} m + (2^{L-\ell} - 1) C_h + (C_h - C_g) R(n), \quad (8)$$

$$\bar{\omega} = 2^{\ell-L} [GC^{-1}(n) + 0.5], \quad (9)$$

where  $C_h$  and  $C_g$  are respectively the energy centers of the low-pass and high-pass quadrature filters [7],  $R(n)$  is an integer obtained by bit reversal of  $n$  in a  $L - \ell$  bits binary representation, and  $GC^{-1}$  is the inverse Gray code permutation. The width and height of the tile are given by

$$\Delta t = 2^{-\ell}, \quad \Delta \omega = 2^{\ell-L}. \quad (10)$$

The SIWPD [7] is preferable to the standard wavelet packet decomposition (WPD) [8] due to its enhanced properties. Namely, shift-invariance, lower information cost and improved time-frequency resolution. For example, the signal  $s(t)$ , which is depicted in Fig. 1, comprises a short pulse, a tone and a component with nonlinear frequency modulation. Its optimal expansions obtained by the Matching Pursuit [9], Basis Pursuit [10] and WPD are illustrated in Fig. 2. While these algorithms use the conventional library of wavelet packets and fail to represent the signal efficiently, the SIWPD (Fig. 2(d)) concentrates the signal into a small number of coefficients. Furthermore, its computational complexity is significantly lower than those associated with the Matching Pursuit and the Basis Pursuit.

Fig. 3 illustrates the MWD for  $g(t)$ , using various distance-thresholds. When  $D = 0$ , there are no interference

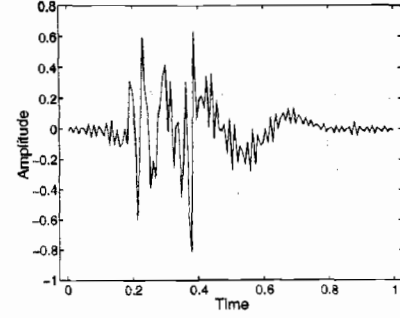


Figure 1. Test signal  $s(t)$  consisting of a short pulse, a tone and a nonlinear chirp.

terms, but the energy concentration of individual components is insufficient.  $D = 2$  leads to improved energy concentration, yet, no significant interference terms are present. As  $D$  gets larger, the interference between components becomes visible and the MWD converges to the conventional WD. An acceptable compromise is usually found between  $D = 1.5$  and  $D = 2.5$  [11].

To illustrate the performance of the MWD, we refer to the mesh plots for the signal  $s(t)$ , which are depicted in Fig. 4. Compared with the Smoothed pseudo Wigner distribution, the Choi-Williams distribution, the cone-kernel distribution and the reduced interference distribution, the SIWPD-based MWD obtains high resolution and concentration in time-frequency, and is superior in eliminating interference terms associated with the WD.

### 3. GENERAL PROPERTIES

**Realness:** The MWD is always real, even if the signal or the basis functions are complex. This property is a direct consequence of the realness of the Wigner distribution.

**Shift-Invariance:** Shifting a signal by  $\tau = k \cdot 2^{-L}$  ( $k, L \in \mathbb{Z}$ ), where  $L$  is finest resolution level of the best-basis decomposition, entails an identical shift of the MWD, *i.e.*, if  $g(t) = g(t - \tau)$  then  $T_g(t, \omega) = T_g(t - \tau, \omega)$ .

**Symmetry in Frequency:** Real signals have symmetrical spectra. For symmetric spectra, the Wigner distribution is symmetric in the frequency domain,  $W_g(t, -\omega) = W_g(t, \omega)$ ,  $W_{g,s}(t, -\omega) = W_{s,g}(t, \omega)$ . Thus, for real signals and real basis-functions, the MWD retains the same symmetries, *i.e.*,  $T_g(t, -\omega) = T_g(t, \omega)$ .

**Symmetry in Time:** For symmetrical signals, the Wigner distribution is symmetric in the time domain,  $W_g(-t, \omega) = W_g(t, \omega)$ ,  $W_{g,s}(-t, \omega) = W_{s,g}(t, \omega)$ . However, the MWD is not necessarily symmetric, since the SIWPD is generally asymmetric. Still, confining ourselves to symmetric basis-functions (entailing either biorthogonal or complex-valued basis-functions) and restricting the SIWPD to bases satisfying  $\{\varphi_\lambda(t)\}_\lambda = \{\varphi_\lambda(-t)\}_\lambda$ , the expansion becomes symmetric, rather than shift-invariant. In that case, the MWD is symmetric in time, *i.e.*,  $T_g(-t, \omega) = T_g(t, \omega)$ .

**Total Energy:** Integrating the general form of the MWD with respect to time and frequency shows that the total energy is bounded by the energy of the signal:

$$\frac{1}{2\pi} \int dt \int d\omega T_g(t, \omega) = \sum_{\lambda \in \Lambda} |c_\lambda|^2 \leq \sum_\lambda |c_\lambda|^2 = \|g\|^2.$$

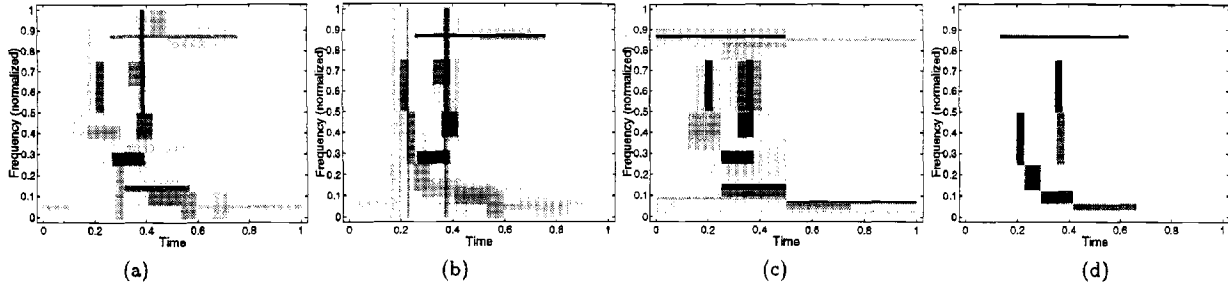


Figure 2. Time-frequency tilings for the signal  $s(t)$  obtained by various best-basis methods: (a) Matching Pursuit; (b) Basis Pursuit; (c) Wavelet Packet Decomposition; (d) Shift-Invariant Wavelet Packet Decomposition.

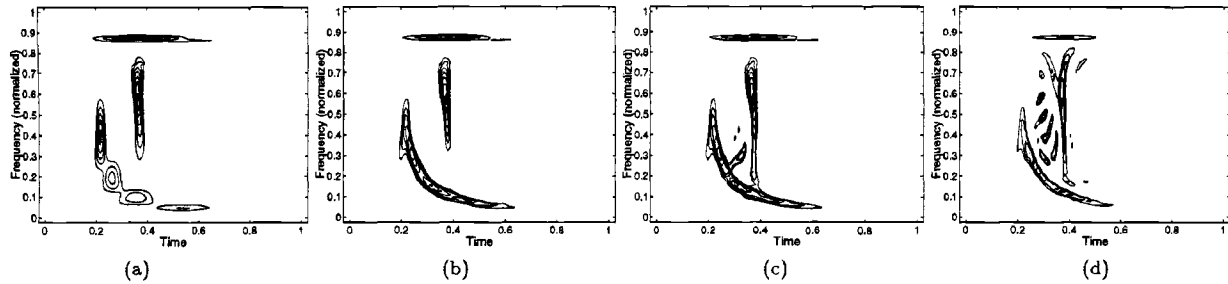


Figure 3. The modified Wigner distribution for the signal  $s(t)$ , combined with the SIWPD and various distance-thresholds: (a)  $D = 0$ ; (b)  $D = 2$ ; (c)  $D = 3$ ; (d)  $D = 5$ . An acceptable compromise between energy concentration and suppression of interference components is usually found for  $1.5 \leq D \leq 2.5$ .

The difference between the total energy and the energy of the signal essentially stems from the smallest expansion coefficients. In fact, if we set the amplitude threshold ( $\epsilon$ ) to zero, the set of indices  $\Lambda$  runs over all the basis-functions, and thus the total energy equals the energy of the signal.

**Positivity:** The interpretation of the conventional WD as a pointwise time-frequency energy density is generally restricted by the uncertainty principle and by the fact that the WD may locally assume negative values. However, the non-negativity and interference terms are closely related, and in many cases the suppression of interference terms accompanies reduction of negative values in magnitude [1]. Thus, reduction of the interference terms associated with the WD, entails comparable attenuation of its negative values.

#### 4. INVERSION AND UNIQUENESS

One of the advantages of the MWD is its capability to resolve multicomponent signals into disjoint time-frequency regions.

##### Definition 1

Let  $X = \Lambda \cup \{\lambda \mid \{\lambda, \lambda'\} \in \Gamma \text{ for some } \lambda' \in \Lambda\}$  be the indices set of the significant basis functions, i.e., the basis functions which contribute to the MWD. A pair of indices  $k, \ell \in X$  are said to be equivalent, denoted by  $k \sim \ell$ , if  $k \equiv \ell$  or alternatively there exists a finite series  $\{\lambda_i\}_{i=1}^N$  such that  $\{\lambda_i, \lambda_{i+1}\} \in \Gamma$  for  $i = 1, 2, \dots, N-1$  and  $\{k, \lambda_1\}, \{\ell, \lambda_N\} \in \Gamma$ .

Clearly,  $\sim$  is an equivalence relation on  $X$ , since it is reflexive ( $k \sim k$  for all  $k \in X$ ) symmetric ( $k \sim \ell$  implies  $\ell \sim k$ ) and transitive ( $k \sim \ell$  and  $\ell \sim m$  imply  $k \sim m$ ). The equivalence relation means that the corresponding basis-functions are linked in the time-frequency plane by a series

of consecutive adjacent basis-functions. Denote by

$$\Lambda_k = \{\lambda \in X \mid \lambda \sim k\} \quad (11)$$

the equivalence class for  $k \in X$ . Then, for any  $k, \ell \in X$  either  $\Lambda_k = \Lambda_\ell$  or  $\Lambda_k \cap \Lambda_\ell = \emptyset$ . Hence,  $\{\Lambda_k \mid k \in X\}$  forms a partition of  $X$ , and each equivalence class can be related to a single component of the signal. The number of components comprising the signal is determined by the number of distinct equivalence classes in  $X$ .

**Lemma 1** Let  $\{\varphi_k\}_{k \in \mathbb{N}}$  be the best basis for  $g(t)$ , and let  $W_{k,\ell} \equiv W_{\varphi_k, \varphi_\ell}$  be the cross Wigner distribution of pairs of basis-functions. Then the set  $\{W_{k,\ell}\}_{k,\ell \in \mathbb{N}}$  is an orthonormal basis for  $L_2(\mathbb{R}^2)$ , and the expansion coefficients for the MWD are given by

$$c_{k,\ell} = \langle T_g, W_{k,\ell} \rangle = \begin{cases} c_k c_\ell^*, & \text{if } k = \ell \in \Lambda \text{ or } \{k, \ell\} \in \Gamma, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

where

$$\langle T_g, W_{k,\ell} \rangle \triangleq \frac{1}{2\pi} \iint T_g(t, \omega) W_{k,\ell}^*(t, \omega) dt d\omega.$$

The proof is detailed in [11]. The components of a prescribed signal can be recovered from the MWD to within an arbitrary constant phase factor in each signal component, and to within errors generated by neglecting small basis constituents (small auto-terms, small cross-terms, as well as interference terms that correspond to distant basis functions). Let  $k \in \Lambda$ , and let  $\Lambda_k$  be its equivalence class. Then for any  $\ell \in \Lambda_k$  there exists a finite series  $\{\lambda_i\}_{i=1}^N$  such that

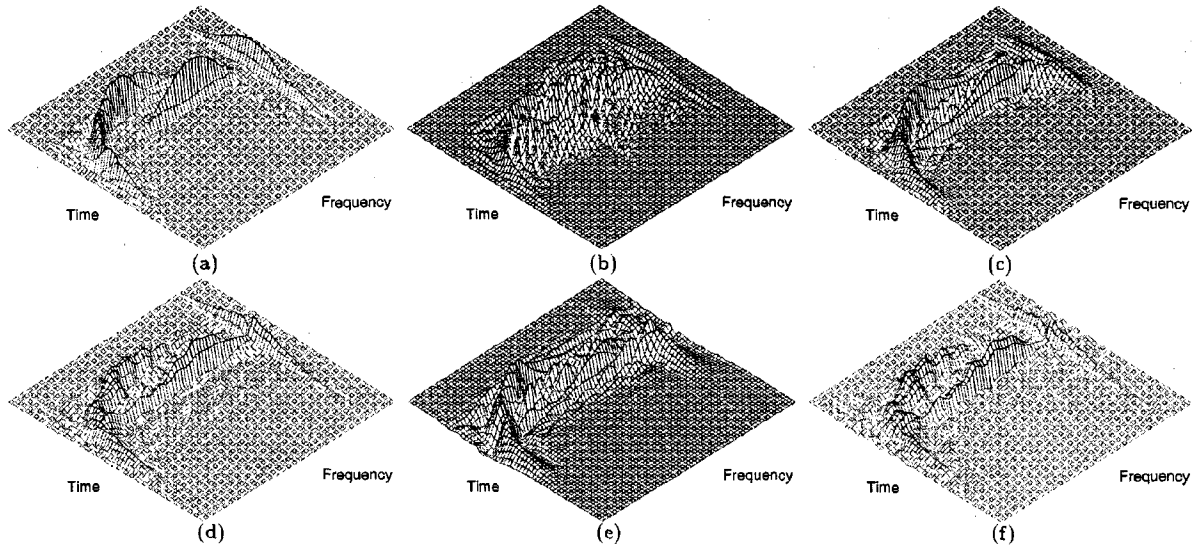


Figure 4. Mesh plots for the signal  $s(t)$ : (a) The modified Wigner distribution combined with the SIWPD and distance-threshold  $D = 2$ ; (b) Wigner distribution; (c) Smoothed pseudo Wigner distribution; (d) Choi-Williams distribution; (e) Cone-kernel distribution; (f) Reduced interference distribution.

$\{\lambda_i, \lambda_{i+1}\} \in \Gamma$  for  $i = 1, \dots, N-1$  and  $\{k, \lambda_1\}, \{\ell, \lambda_N\} \in \Gamma$ . By Eq. (12) we have

$$|c_k|^2 = \langle T_g, W_{k,k} \rangle, \quad (13)$$

$$c_k c_{\lambda_1}^* = \langle T_g, W_{k,\lambda_1} \rangle, \quad (14)$$

$$c_{\lambda_i} c_{\lambda_{i+1}}^* = \langle T_g, W_{\lambda_i, \lambda_{i+1}} \rangle, \quad i = 1, \dots, N-1, \quad (15)$$

$$c_{\lambda_N} c_\ell^* = \langle T_g, W_{\lambda_N, \ell} \rangle, \quad (16)$$

which shows that  $c_\ell$  has a recursive relation to  $c_k$ , and  $c_k$  can be recovered from the MWD up to a phase factor. Accordingly, each component of the signal can also be recovered up to an arbitrary constant phase factor by  $g_k = \sum_{\ell \in \Lambda_k} c_\ell \varphi_\ell$ . The constant phase factor in each component of the signal clearly drops out when we calculate the MWD (as it does for the WD). Therefore, it cannot be recovered. Summation of distinct signal components generally yields a different signal that has the same MWD. In some applications, such as pattern recognition, it is actually desirable that signals consisting of the same components will be identified, irrespective of their relative phase. The MWD provides an efficient technique for doing so.

## 5. CONCLUSION

Herein we investigate adaptive decompositions leading to a newly defined modified Wigner space. We have shown the validity of various useful properties relevant to time-frequency analysis. Interference terms between distinct components can be efficiently eliminated, as long as the localization properties of basis elements aptly resemble that of the signal. The MWD is shown to be effective for resolving multicomponent signals. The signal components are determined as partial sums of basis-functions over equivalence classes defined in the time-frequency plane.

The proposed methodology is extendable to other distributions (e.g., Cohen's class) and other "best-basis" decompositions. However, the properties of the resulting modified forms depend on the distribution, library of bases and best-basis search algorithm which are specifically employed.

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