CCIT Report #411 January 2003

Bounds on the Performance of Vector-Quantizers under Channel Errors

Gal Ben-David and David Malah
Technion - Israel Institute of Technology
Dept. of Electrical Eng. Haifa 32000, Israel

Correspondence: Gal Ben-David, Mitzpe Adi 238, D.N. Nazareth Illit 17940, Israel
Tel: + 972-4-9860 398, Fax: + 972-4-9860 401, Cell: +972-54-698 289
e-mail: galb@netvision.net.il , malah@ee.technion.ac.il

Abstract

Vector-Quantization (VQ) is an effective and widely known method for low-bit-rate communication of speech and image signals. A common assumption in the design of VQ-based communication systems is that the compressed digital information is transmitted through a perfect channel. Under this assumption, quantization distortion is the only factor in output signal fidelity. Moreover, the assignment of channel symbols to the VQ reconstruction vectors is of no importance. However, under physical channels, errors may be present, causing degradation in overall system performance. In such a case, the effect of channel errors on the coding system performance depends on the index assignment of the reconstruction vectors. The index assignment problem is a special case of the Quadrature Assignment Problem and is known to be NP-complete. For a VQ with N reconstruction vectors there are $N!$ possible assignments, meaning that an exhaustive search over all possible assignments is practically impossible. To help the VQ designer, we present in this paper lower and upper bounds on the performance of VQ systems under channel errors, over all possible assignments. These bounds are based on eigenvalue arguments and perform better than general bounds for the Quadrature Assignment Problem. A related expression for the average performance is also given and discussed. Special cases and numerical examples are given in which the bounds and average performance are compared with index assignments obtained by a well-known index-switching algorithm.

Index Terms: Vector-quantization, index assignment, channel coding, performance bounds

I. Introduction

Vector Quantization (VQ) is a method for mapping signals into digital sequences. A typical VQ-based communication system is shown in Fig. 1.

A discrete-time *source* emits signal samples over an infinite (or densely finite) alphabet. These samples should be sent to the *destination* with the highest possible fidelity. The *VQ encoder* translates source output vectors into *channel* digital sequences. The *VQ decoder*'s goal is to reconstruct source samples from this digital information. Since the analog information cannot be perfectly represented by the digital information some *quantization distortion* must be tolerated.

In each channel transmission the VQ encodes a *K*-dimensional vector of source samples - $\underline{x}(t)$ into a *reconstruction vector index* $y(t)$, where the discrete variable *t* represents the time instant or a channel-use counter. The index is taken from a finite alphabet, $y(t) \in \{0,1,\ldots,N-1\}$, where N is the number of reconstruction vectors (hence the number of possible channel symbols).

The *Index Assignment* (IA) is represented in Fig. 1 by a permutation operator:

$$\Pi : \quad y(t) \in \{0,1,\ldots,N-1\} \rightarrow z(t) \in \{0,1,\ldots,N-1\} \tag{1}$$

The number of possible permutations, $N!$, increases very fast with *N*. E.g., for a VQ with just 4-bits index representation ($N = 16$ indices) there are $16! \approx 2 \cdot 10^{13}$ possible permutations. For typical values of N, examination of all possible permutations is therefore impractical. The channel index $z(t) = \Pi\{y(t)\}$ is sent through the channel.

For *memoryless channels*, the channel output $\hat{z}(t)$ is a random mapping of its input $z(t)$, characterized by the *channel probability matrix* H, defined by:

$$\{H\}_{ij} = \text{Prob}\{\hat{z}(t) = j | z(t) = i\} \tag{2}$$

Throughout, we assume that H is symmetric (i.e., we consider memoryless channels with a symmetric transition matrix).

For the special case of the *Binary-Symmetric-Channel* (BSC):

$$\{H\}_{ij} = \text{Prob}\{\hat{z}(t) = j | z(t) = i\} = q^{d_H(i,j)}(1-q)^{L-d_H(i,j)} \tag{3}$$

where L is the number of bits ($N = 2^L$) per channel use, q is the *Bit-Error-Rate* (BER) and $d_H(i,j)$ is the *Hamming distance* between the binary representations of i and j.

At the receiver, after inverse-permutation, the *VQ decoder* converts the channel output symbols into one of N possible reconstruction vectors. The decoder's output $\underline{\hat{x}}(t)$ is, hopefully, "close" to the original input. The term "close" will be defined by a distortion measure, $d(\underline{x},\underline{\hat{x}})$, between the input and the output of the VQ system.

Knowledge of the source statistics $p(\underline{x})$ or the availability of a representing *training sequence* is assumed. The performance of the overall system is measured in terms of the average distortion $E[d(\underline{x},\underline{\hat{x}})]$.

In "classic" discussions of VQ applications, the channel is assumed to be noiseless ($H = I$, where I is the unity matrix) [1], so that no errors occur during transmission and $y(t) = \hat{y}(t)$ for every t. This assumption is based upon using a channel encoder-decoder pair to correct channel errors, causing the distortion due to channel-errors to be negligible. The permutation Π has no effect in this case.

Upon knowledge of the source statistics, Lloyd's algorithm [1] may be used to design the VQ. In practice, a training sequence is used and the LBG algorithm [1] is implemented. Both methods are iterative and alternately apply the *nearest-neighbor* condition and the *centroid* condition.

In some applications, channel coding is not utilized due to its complexity or because of bit-rate constraints. In case of a channel error event, a wrong reconstruction vector is selected at the decoder. The distortion due to channel errors can be significant and affects the design of the VQ system [2-16].

In the literature two main approaches are proposed to improve the performance of vector quantizers under channel errors. The first method allows modification of the partition regions and their corresponding codevectors. In the presence of channel errors, and given the transmitted symbol, the received symbol is a random variable. It is suggested to redesign the VQ by modifying the distortion measure to take all possible output vectors into consideration. This modification results in a *weighted-nearest-neighbor* and *weighted-centroid* conditions [7],[8],[10],[27]. These conditions are specific to every channel condition. Hence, a VQ designed for a noisy channel should, in principle, monitor channel conditions, and apply a different partition and a different set of codevectors for each possible BER. Other drawbacks of this approach are the large memory consumption and extensive design effort. A simple suboptimal method is suggested in [4], where a linear mapping is used for the modification of the partition regions.

The second approach tries to reduce channel distortion by using better index assignments. The search for the optimal index assignment is a special case of the Quadratic-Assignment Problem (QAP) and is known to be NP-Complete [9].

Several suboptimal methods are suggested in the literature. In [11], [12] an iterative algorithm is proposed. After selecting an initial assignment, the algorithm searches for a better assignment by exchanging indices of codevectors, and keeping the new assignment if it performs better than its predecessor. This algorithm can only offer a local minimum. A more sophisticated algorithm is examined in [8], where Simulated Annealing (SA) is used to search for an optimal index assignment. The method of SA involves some ad-hoc arguments to define system "temperature" and "cooling" procedures. Moreover, the method of SA has a very slow convergence rate, and cannot assure global optimum during a limited design period. A sub-optimal quadratic placement algorithm [18] is used in [19] for obtaining an efficient VQ index assignment. Implementation of a search approach for quadratic assignment problems, known as Tabu, is examined in [20]. The Tabu search explores the entire set of possible index assignments by a sequence of moves. Keeping visited assignments in a dedicated memory prevents cycling. Similar to the SA, the Tabu search has a slow convergence rate and cannot assure a global optimum.

For the special case of a uniform scalar quantizer and a uniform source under the binary symmetric channel (BSC), it is shown in [25] that the natural binary code (NBC) assignment is an optimal assignment. Later approaches, using eigenvalue arguments, [3], [6] have reached the same conclusion. The NBC is also optimal for the 4-bit uniform scalar quantizer using a (4,7) Hamming error control code, under the BSC channel [3].

The difficulty in obtaining good assignments validates our development of performance bounds. These bounds and a related expression for the average performance over all possible index assignments may benefit the VQ designer in estimating the performance of a given assignment. Given a VQ structure, upper and lower bounds on the "assignment gain" benefit the VQ designer searching for an efficient assignment. The evaluation of the average performance, over all index assignments, can also help in revealing how well a given assignment performs.

The remainder of the paper is organized as follows. In section II, the distortion due to channel errors is defined. The optimization of the channel distortion over all possible index assignments is discussed. In section III, bounds on the performance of a given VQ system under a given symmetric and memoryless channel, over all possible index assignments, are obtained. The bounds are shown to provide better performance than more general QAP bounds. A related expression for the average performance over all index assignments is presented in section IV. Special cases and numerical results, obtained in simulations, are presented and discussed in section V, while conclusions are given in section VI.

II. Channel Distortion

A vector quantization system is characterized by a set of codevectors and a corresponding partition of the signal space R of all possible input vectors - $\underline{x}$. This space is partitioned into N regions, R_i, $i = 0,1,\ldots,N-1$. These regions cover the whole signal space and are nonoverlapping:

$$\begin{aligned} &\bigcup_i R_i = R \\ &R_i \cap R_j = \varnothing \end{aligned} \tag{4}$$

Each partition region R_i has a corresponding *reconstruction (*or *representation) vector* - $\underline{\phi}_i$. For the special case of *centroid quantizers*: $\underline{\phi}_i = \int_{R_i} \underline{x} \cdot p(\underline{x}) \cdot d\underline{x} \Big/ \int_{R_i} p(\underline{x}) \cdot d\underline{x}$.

The encoder accumulates a *K*-dimensional vector of source samples $\underline{x}$. The symbol $y(t) = i$ is emitted if $\underline{x} \in R_i$. The corresponding channel symbol, $z(t) = \Pi(i)$, is transmitted through the channel. The channel's output is a random mapping of this transmission. Upon receiving the channel symbol $\hat{z}(t) = j$ the decoder emits the reconstruction vector that corresponds to the index $\Pi^{-1}(j)$.

The overall distortion of the VQ-based communication system is:

$$D_T = E\left[d\left(\underline{x},\hat{\underline{x}}\right)\right] = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left\{ \pi \cdot H \cdot \pi^T \right\}_{ij} \int_{R_i} d\left(\underline{x},\underline{\phi}_j\right) \cdot p(\underline{x}) \cdot d\underline{x} \tag{5}$$

In (5) the permutation is represented by a permutation matrix - π, whose entries are 0's and 1's and the sum of elements in each of its rows and columns is 1. The permutation matrix is self-orthogonal, i.e., $\pi\pi^T = I$.

For the perfect channel, $H = I$, the permutation matrix π is of no importance, and the only factor affecting system performance is the *quantization distortion*:

$$D_T\big|_{H=I} = E\left[d\left(\underline{x},\hat{\underline{x}}\right)\right]_{H=I} = \sum_{i=0}^{N-1}\int_{R_i} d\left(\underline{x},\underline{\phi}_i\right)\cdot p\left(\underline{x}\right)\cdot d\underline{x} \tag{6}$$

In the following analysis, two distortion terms, the *quantization distortion* D_Q and the *channel distortion* D_C, are defined by:

$$\begin{aligned} D_Q &= \sum_{i=0}^{N-1}\int_{R_i} d\left(\underline{x},\underline{\phi}_i\right)\cdot p\left(\underline{x}\right)\cdot d\underline{x} \\ D_C &= \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1}\left\{\pi\cdot H\cdot \pi^T\right\}_{ij}\cdot d\left(\underline{\phi}_i,\underline{\phi}_j\right) = trace\left\{P\cdot\pi\cdot H\cdot\pi^T\cdot D\right\} = \\ &= trace\left\{D\cdot P\cdot\pi\cdot H\cdot\pi^T\right\} \quad , \end{aligned} \tag{7}$$

where p_i is the probability of $\underline{x}\in R_i$:

$$p_i = \int_{R_i} p\left(\underline{x}\right)\cdot d\underline{x} \tag{8}$$

The matrix P in (7) is a diagonal matrix, which contains these probabilities, i.e., $P = diag\left\{p_0, p_1, \ldots, p_{N-1}\right\}$, and the entries of the matrix D are the distances between all possible pairs of reconstruction vectors: $D_{ij} = d\left(\underline{\phi}_i,\underline{\phi}_j\right)$.

It is shown in [8], [10] that for the squared Euclidean distance measure and centroid quantizers, the overall distortion is the sum of the quantization and channel distortions: $D_T = D_Q + D_C$. This result is also applicable for quantizers with a large number of codevectors ($N\to\infty$) [27].

For the special case of a uniform *scalar* quantizer with a quantization step h and a uniform source, the codevectors are the scalars $\phi_i = \left(i - N/2\right)\cdot h$; distances are $d\left(\phi_i,\phi_j\right) = h^2\cdot\left(i-j\right)^2$, and the probabilities are $p_i = 1/N,\ i = 0,1,\ldots,N-1$. In this case, and transmission over a binary symmetric channel, it is shown in [25] that the natural binary code (NBC) assignment, corresponding here to $\pi = I$, is an optimal assignment. Using eigenvalue arguments, later approaches [3], [6] have obtained the same result.

III. Performance bounds

Minimization of channel distortion, as defined in (7), over all possible index assignments (or permutation matrices - π) is known to be a special case of the QAP – quadratic assignment problem [7]. The complexity of the QAP is known to be NP-complete and therefore obtaining optimal assignments may not be feasible.

A simple bound of the QAP is given in [22], under theorem 2.1:

$$\sum_{i=0}^{N-1} \lambda_i(D \cdot P) \cdot \lambda_{N-i}(H) \le D_C = trace\{D \cdot P \cdot \pi \cdot H \cdot \pi^T\} \le \sum_{i=0}^{N-1} \lambda_i(D \cdot P) \cdot \lambda_i(H) \tag{9}$$

where $\lambda(M)$ is a vector containing the eigenvalues of the matrix *M*, in non-decreasing order.

Note that the main diagonal of the matrix DP is all zeros and the sum of its eigenvalues is therefore zero. Now, the lower bound is the inner product between the eigenvalues of DP, ordered from the most negative to the most positive and the eigenvalues of *H,* ordered in non-increasing order. The lower bound is therefore zero (if all the eigenvalues of *H* have the same value), or negative, thus not providing new information.

In this section we introduce lower and upper bounds on the channel distortion D_C, under memoryless channels with a symmetric transition matrix, over all possible assignments (or permutation matrices - π). The derivation of the bounds is related to the QAP projection technique presented in [22]. Nevertheless, by using some special properties of the matrices involved in the index assignment problem we gain tighter bounds.

As in [5], [16], we define a symmetric matrix $\hat{D}$ as:

$$\hat{D} = DP + P^T D^T , \tag{10}$$

So that by using the symmetry property of the channel matrix, H, the channel distortion becomes:

$$D_C = \frac{1}{2} trace\left\{ H\pi^T \hat{D}\pi \right\} \tag{11}$$

The bounding technique is based on eigenvalues arguments. Instead of optimizing over the (discrete) family of matrices covering all possible assignments π, we optimize over a wider (continuous) family. Moreover, special properties of the channel transition matrix Q are used to obtain upper and lower bounds that are tighter than known bounds [9][15].

A fundamental step in this optimization procedure is to replace the matrix $\hat{D}$, defined in (10), by another symmetric matrix $\widetilde{D}$, such that the all-ones vector $\underline{1} = [1 \quad 1 \quad \cdots \quad 1]^T$ is its eigenvector, while D_C is just changed by a known constant. This goal is achieved by the following procedure:

First, we denote a matrix of the form shown in (12) as a “column structured” matrix:

$$C_i = \begin{array}{c} \begin{bmatrix} 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \\ 0 & & 0 & 1 & 0 & & & 0 \end{bmatrix} \\ \uparrow i - \text{th column} \end{array} = \underline{1} \cdot \underline{e}_i^T \tag{12}$$

where $\underline{e}_i^T = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ and the 1 is located at the *i*-th location. Recalling that H represents probabilities, the sum of elements in any of its rows is one, so the vector $\underline{1}$ is an eigenvector of H: $H \cdot \underline{1} = \underline{1}$. The same argument is valid for the *i*-th column of the matrix C_i: $H \cdot C_i = C_i$, $i = 0,1,\ldots,N-1$.

We now construct a symmetric matrix $\alpha\left(C_i + C_i^T\right)$, where α is a scalar, denoted here as a *cross structured matrix*. It is simple to show that, regardless of the permutation matrix π, adding a *cross structured* matrix to the matrix $\hat{D}$ changes the expression in (11) just by the addition of the scalar α:

$$\frac{1}{2} trace\left\{ H \pi^T \left[\hat{D} + \alpha\left(C_i^T + C_i \right) \right] \pi \right\} = \frac{1}{2} trace\left\{ H \pi^T \hat{D} \pi \right\} + \alpha \tag{13}$$

Let s_i denote the sum of the elements in the *i*-th row of the matrix $\hat{D}$

$$s_i = \sum_{j=0}^{N-1} \hat{D}_{ij} \,, \tag{14}$$

and let k denote any one of the indices of lines having the largest sum (i.e., $s_k \geq s_i \quad i = 0,1,\ldots,N-1$).

In order to achieve the desired property $\widetilde{D}\cdot\underline{1}=\omega_0\underline{1}$, for some ω_0, all rows of $\widetilde{D}$ must have the same sum of entries. Let us examine the effect of adding the "*cross structured"* matrix $\alpha\left(C_i+C_i^T\right)$ to a general matrix M of size $N\times N$. The sum of elements in all rows except for the *i*-th row is increased by α, while the sum of elements in the *i*-th row is increased by $(N+1)\cdot\alpha$.

Therefore, we define the matrix $\widetilde{D}$ to be

$$\widetilde{D}=\hat{D}+\sum_{i=0}^{N-1}\alpha_i\left(C_i+C_i^T\right), \tag{15}$$

where,

$$\alpha_i=\left(s_k-s_i\right)/N. \tag{16}$$

Having $s_k\geq s_i$ by the selection of *k*, the scalars α_i are all non-negative. By adding *N*-1 *cross structured* matrices (if $s_i=s_k$ it is the zero matrix) we get that $\widetilde{D}$ is a symmetric matrix with all its rows having the same sum of elements, and with the desired property $\widetilde{D}\cdot\underline{1}=\omega_0\underline{1}$. We shall refer to $\widetilde{D}$ as the *weighted distance matrix*. The channel distortion can now be written as:

$$D_C=\frac{1}{2}trace\left\{H\pi^T\widetilde{D}\pi\right\}-S \tag{17}$$

where

$$S=\sum_{i=0}^{N-1}\alpha_i. \tag{18}$$

At this point, it is interesting to note that both the channel matrix H and the weighted distance matrix $\widetilde{D}$ are symmetric, have nonnegative entries, and have the vector $\underline{1}=\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ as an eigenvector. Moreover, because of the symmetry property, all eigenvalues of both matrices are real.

Next, we use the following theorem adopted from [21 Section 15.7].

Theorem: Given a symmetric matrix M with nonnegative-entries having the property $M \cdot \underline{1} = \beta \cdot \underline{1}$, the eigenvalue β (known as the *Perron-Frobenius eigenvalue*) is positive and is the largest eigenvalue of M in absolute value (there may be negative eigenvalues, but smaller in absolute value).

Corollary: The eigenvalue 1 of the matrix H and the eigenvalue $\omega_0 > 0$ of the matrix $\widetilde{D}$, both corresponding to the eigenvector $\underline{1}$, are each the largest eigenvalue in absolute value of the corresponding matrix.

Next, we use eigenvalue arguments to obtain bounds the channel distortion. We perform a unitary diagonalization on both matrices:

$$\begin{aligned} H &= V \cdot \Lambda \cdot V^T , \quad V \cdot V^T = I \\ \widetilde{D} &= W \cdot \Omega \cdot W^T , \quad W \cdot W^T = I \end{aligned} \tag{19}$$

Without loss of generality, we sort the eigenvalues (and corresponding eigenvectors) in Λ and Ω in decreasing order. Substituting (19) into (17):

$$\begin{aligned} D_C &= \frac{1}{2} trace\left\{ V\Lambda V^T \cdot \pi^T W\Omega W^T \pi \right\} - S = \\ &= \frac{1}{2} trace\left\{ \Lambda V^T \pi^T W\Omega W^T \pi V \right\} - S = \\ &= \frac{1}{2} trace\left\{ \Lambda \Psi \Omega \Psi^T \right\} - S = \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \lambda_i \omega_j \psi_{ij}^2 - S , \end{aligned} \tag{20}$$

where we define $\lambda_i = \Lambda_{ii}, \omega_i = \Omega_{ii}, i = 0,1,\ldots,N-1$ and the matrix Ψ is defined as $\Psi = V^T \pi^T W$. The matrix Ψ is also unitary since $\Psi\Psi^T = V^T \pi^T W \cdot W^T \pi V = I$.

For the special case of an L bit binary word transmitted trough a binary-symmetric-channel, the eigenvalues λ_i, $i = 0,1,\ldots,N-1$, are calculated in [6]. There are $L+1$ distinct eigenvalues, $(1-2q)^m$, $m = 0,1,\ldots,L$, each with multiplicity $\binom{L}{m}$, where q is the bit-error-rate.

Observe that since the first column of both V and W is $\underline{v}_0 = \underline{w}_0 = \frac{1}{\sqrt{N}}\underline{1}$, the sum of elements in the remaining columns of both matrices is zero. The structure of $\Psi = V^T \pi^T W$ is therefore:

$$\Psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & ? & \\ 0 & & & \end{bmatrix} \tag{21}$$

where the question mark represents unknown entries.

In order to obtain upper and lower bounds over all possible index assignments, we relax the constraint that the matrix Ψ in (20) equals to $V^T \pi^T W$ for some permutation matrix - π. This method is known as an orthogonal relaxation of symmetric QAP [22]. The relaxation is done in two steps. In the first step we replace the discrete family of matrices Ψ by the continuous family of unitary matrices having a general structure as in (21). In the second step we replace the unitary requirement by a more relaxed condition. We merely demand that the sum of squares of the elements in each row and column is 1. We shall show that the second relaxation still results in a unitary matrix and hence does not degrade the tightness of the bounds.

In order to obtain the extreme values of the relaxed problem, we state the following optimization problem, using the property that the sum of squares of the elements in each row and column of a unitary matrix (Ψ in this case, with elements ψ_{ij}) is equal to 1:

$$\begin{aligned}
&\min_{\Psi}/\max_{\Psi}\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j\psi_{ij}^2\right)\\
&\text{subject to } \sum_{i=1}^{N-1}\psi_{ij}^2=1,\quad j=1,2,\ldots,N-1 \qquad (22)\\
&\qquad\qquad\quad \sum_{j=1}^{N-1}\psi_{ij}^2=1,\quad i=1,2,\ldots,N-1
\end{aligned}$$

Note that the first row (*i=0*) and the first column (*j=0*) are independent of the permutation and were omitted from the optimization problem. We denote the solutions for the minimum/maximum problems by $\Psi_{\min}/\Psi_{\max}$, respectively. The solution of the optimization problems is given in [22] :

$$\begin{array}{ll}
\text{Minimum value}: \sum_{i=1}^{N-1}\lambda_i\cdot\omega_{N-i} & \text{Maximum value}: \sum_{i=1}^{N-1}\lambda_i\cdot\omega_i \\
\text{Corresponding to}: & \text{Corresponding to}: \\
\Psi_{\min}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & & & \pm1\\ \vdots & & & \pm1 & \\ 0 & & \cdot^{\cdot^{\cdot}} & & \\ 0 & \pm1 & & & 0\end{bmatrix} & \Psi_{\max}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0\\ 0 & \pm1 & & & 0\\ \vdots & & \ddots & & \\ 0 & & & \pm1 & \\ 0 & 0 & & & \pm1\end{bmatrix}
\end{array} \qquad (23)$$

Note that the matrices $\Psi_{\min}$ and $\Psi_{\max}$ are unitary. This implies that the second relaxation did not worsen bounds tightness.

Applying these solutions to (20), the bounds on the channel distortion, over all possible index assignments, are therefore:

$$\frac{1}{2}\lambda_0\omega_0 - S + \frac{1}{2}\sum_{i=1}^{N-1}\lambda_i \cdot \omega_{N-i} \le D_C \le \frac{1}{2}\lambda_0\omega_0 - S + \frac{1}{2}\sum_{i=1}^{N-1}\lambda_i \cdot \omega_i \qquad (24)$$

Another representation of the bounds may be obtained by using the fact that $\lambda_0 = 1$ and, as may be seen from (15) and (16), $\omega_0 = S + s_k$:

$$\frac{1}{2N}\underline{1}^T \cdot \hat{D} \cdot \underline{1} + \frac{1}{2}\sum_{i=1}^{N-1}\lambda_i \cdot \omega_{N-i} \le D_C \le \frac{1}{2N}\underline{1}^T \cdot \hat{D} \cdot \underline{1} + \frac{1}{2}\sum_{i=1}^{N-1}\lambda_i \cdot \omega_i \qquad (25)$$

The first part of both inequalities is independent of the channel. The second part depends on the differences between both channel and weighted distance matrix eigenvalues.

In conclusion, in order to find the desired bounds, given the channel transition matrix - *H*, the VQ distance matrix – *D*, and the a-priori probabilities matrix – *P*, one should carry out the following steps:

1. From D and P, Calculate the scalars $s_i, i = 0,1,\ldots,N-1$, using (14), $\alpha_i, i = 0,1,\ldots,N-1$ (16), S (18), and the *weighted distance matrix*, $\widetilde{D}$, using (19).
2. Calculate the eigenvalues of the *channel matrix* H ($\lambda_i, i = 0,1,\ldots,N-1$), and those of the *weighted distance matrix* $\widetilde{D}$ ($\omega_i, i = 0,1,\ldots,N-1$).
3. Calculate the upper and lower bounds using (24) or (25), where in the latter also (10) needs to be applied.

The upper and lower bounds in (24), (25), were obtained by using $2N$ constrains on the sum of squares of elements in each row and column of the unitary matrix $\Psi = V^T \pi^T W$. It is possible to add further linear constraints, thus achieving tighter upper and lower bounds, as we show below.

In general, an unknown entry of the matrix Ψ is an inner product of two vectors:

$$\psi_{ij} = \left(\underline{v}_i, \pi \underline{w}_j\right), \quad i, j = 1,2,\ldots,N-1 \tag{26}$$

where $\underline{v}_i$ is the i-th column of the matrix V and $\underline{w}_j$ is the j-th column of the matrix W, as defined in (19).

As shown in [22], the largest (smallest) value of the inner product is achieved by ordering the entries of the vector $\underline{v}_i$ in the same (reverse) order as the entries of $\underline{w}_j$. We denote these largest and smallest values by ψ_{ij}^L and ψ_{ij}^S respectively. Recalling that both W and V are unitary (the sum of squares of elements in each column is 1), and using the Cauchy-Schwarz inequality, it is clear that $0 \le \left(\psi_{ij}^S\right)^2, \left(\psi_{ij}^L\right)^2 \le 1$. Using this argument, we can restate the optimization problem of (22) as:

$$\begin{aligned}
&\min_{\Psi}/\max_{\Psi}\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j\psi_{ij}^2\right) \\
&\text{subject to } \sum_{i=1}^{N-1}\psi_{ij}^2 = 1, \quad j = 1,2,\ldots,N-1 \\
&\qquad\qquad\quad \sum_{j=1}^{N-1}\psi_{ij}^2 = 1, \quad i = 1,2,\ldots,N-1 \\
&\qquad\qquad\quad 0 \le \psi_{ij}^2 \le \max\left\{\left(\psi_{ij}^S\right)^2, \left(\psi_{ij}^L\right)^2\right\}, \quad i,j = 1,2,\ldots,N-1
\end{aligned} \tag{27}$$

The optimization problem in (27) can be solved *numerically* using linear programming techniques. The lower and upper bounds obtained by solving (27) are tighter than the analytical bounds in (24), (25). However, in our numerical examples, the difference between the two sets of bounds was very small, hence only the analytical bounds in (24), (25), are further considered in this paper.

IV. Average Performance over all Index Assignments

Having found lower and upper bounds on the channel distortion, it is also useful to obtain the average value of the channel distortion over all possible index assignments. The average value can help in ranking a given assignment.

From (20), this average value is given by:

$$\begin{aligned}\langle D_C\rangle &= \frac{1}{2N!}\sum_{\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\lambda_i\omega_j\psi_{ij}^2 - S = \\ &= \frac{1}{2N!}\sum_{\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\lambda_i\omega_j\left[\sum_{k=0}^{N-1}v_{ki}w_{\pi(k)j}\right]^2 - S\,,\end{aligned} \tag{28}$$

where the permutation is denoted by π and $v_{ki}, w_{\pi(k)j}$ are the elements of the matrices V and W, respectively. It may be shown that the ensemble average is:

$$\begin{aligned}\langle D_C\rangle &= \frac{1}{2}\lambda_0\omega_0 - S + \frac{1}{2(N-1)}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j \\ &= \frac{1}{2N}\underline{1}^T\cdot\hat{D}\cdot\underline{1} + \frac{1}{2(N-1)}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j\end{aligned} \tag{29}$$

Note that the average value in (29) corresponds to a matrix Ψ with the following structure:

$$\Psi_{\langle D_C\rangle} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \beta & \beta & \cdots & \beta \\ 0 & \beta & \beta & & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \beta & \beta & & \beta \end{bmatrix} \quad \text{where } \beta = \pm\frac{1}{\sqrt{N-1}} \tag{30}$$

The matrix $\Psi_{\langle D_C\rangle}$ in (30) is not unitary and therefore does not correspond to any valid permutation.

Comparing $\Psi_{\langle D_C\rangle}$ in (30) with the structure of the matrices $\Psi_{\min}$ and $\Psi_{\max}$ that correspond to the lower and upper bounds (23), respectively, one observes that the performance of a specific permutation corresponds to the geometric relations among the columns of the matrix V and the columns of the matrix $\pi^T W$. A permutation that aligns the column of the matrix V with the columns of the matrix $\pi^T W$, in direct (reverse) order, results in "poor" ("good") performance. A permutation that does not align the two sets of columns will typically results in "average" performance. This geometric interpretation can help in obtaining future sub-optimal index assignment algorithms. A sub-optimal algorithm may be based on a permutation matrix π in $\Psi = V^T \pi^T W$ that approximates $\Psi_{\min}$ in (23).

V. Special Cases and Simulation Results

In this section we examine several special cases and compare the lower and upper bounds with the average distortion over all index assignment, as well as with the distortion of assignments that were obtained in simulations. We used the well known index-switching algorithm [11], [12] to obtain "good" and "poor" index assignments (IA). According to this algorithm, after selecting an initial assignment, indices of codevectors are randomly exchanged. When searching for a good (poor) assignment, the new assignment is kept if it performs better (worse) than its predecessor.

A. Special Cases

1. Uniform scalar quantizer and a uniform source under the BSC

We obtain here the bounds in (24) for the case of a uniform scalar quantizer and a uniform source operating under the binary symmetric channel. The distance between two consecutive levels of an L-bit ($N = 2^L$ levels) quantizer is assumed to be $2/N$. The resulting upper and lower bounds, from (24), are:

$$\frac{2(N-1)(N+1)}{3N^2} 2q \le D_C \le \frac{2(N-1)(N+1)}{3N^2}\left[1-(1-2q)^L\right], \tag{31}$$

where q is the bit error rate. It is interesting to see that for small values of q, the ratio between the upper and lower bounds is L.

The lower bound coincides with the performance of the Natural Binary Code, which is an optimal assignment for this case, as shown in [25] and demonstrated in [3],[6]. The worst index assignment for this case is given in [26], resulting in a channel distortion given by:

$$D_{WIA} = q(1-2q)^{L-1} + \frac{2(N-1)(N+1)}{3N^2}\left[1-(1-2q)^{L-1}\right] \tag{32}$$

A numerical comparison between the performance of the worst index assignment and the upper bound (30) reveals that the upper bound is 0.25dB higher for a 4-bit quantizer and only 0.13dB higher for an 8-bit quantizer.

Using (28) for this case, the average distortion over all index assignments is:

$$\langle D_C \rangle = \frac{2(N+1)}{3N}\left[1-(1-q)^L\right] \tag{33}$$

For small values of the bit error rate, $q \to 0$, the ensemble average approaches zero linearly with q.

$$\langle D_C \rangle \approx \frac{2L(N+1)}{3N} q \text{ as } q \to 0 \tag{34}$$

This expression agrees with an asymptotic result given in [8].

2. Maximum entropy vector quantizers under the BSC

For the special case of a maximum entropy quantizer with a quadratic distance measure and the BSC, an asymptotic lower bound is given in [15]:

$$D_C \geq \frac{4q}{N}\sum_{n=0}^{N-1}\phi_n^T\phi_n \quad as \quad q \to 0 \tag{35}$$

where ϕ_n is the representation vector of the n-th partition region. We define $Y = [\phi_0 \quad \phi_1 \quad \cdots \quad \phi_{N-1}]$. We also assume, without loss of generality, that $\sum_{n=0}^{N-1}\phi_n = 0$.

To compare (35) with the proposed lower bound, we note first that the channel matrix eigenvalues, λ_i, $i = 0,1,\ldots,N-1$, calculated for $q \to 0$, are $1-2mq$, $m = 0,1,...,L$, with multiplicity $\binom{L}{m}$. We represent these eigenvalues by $\lambda_i = 1-2m_i q,\ i = 0,1,\ldots,N-1$, such that $m_0 = 0, m_1 = 1,\ldots,m_L = 1, m_{L+1} = 2,\ldots,m_{N-1} = L$. Note that the eigenvalues are sorted in descending order.

In this case the weighted distance matrix (15) is

$$\begin{aligned}
\widetilde{D} &= \frac{2}{N}\left[D + \sum_{i=0}^{N-1}\left(\phi_k^T\phi_k - \phi_i^T\phi_i\right)\cdot\left(C_i + C_i^T\right)\right] = \\
&= \frac{2}{N}\left[\sum_{i=0}^{N-1}\phi_i^T\phi_i\cdot\left(C_i + C_i^T\right) - 2Y^TY + \sum_{i=0}^{N-1}\left(\phi_k^T\phi_k - \phi_i^T\phi_i\right)\cdot\left(C_i + C_i^T\right)\right] = \\
&= \frac{2}{N}\left[\phi_k^T\phi_k\cdot\sum_{i=0}^{N-1}\left(C_i + C_i^T\right) - 2Y^TY\right] = \\
&= \frac{4\phi_k^T\phi_k}{N}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} - \frac{4}{N}Y^TY
\end{aligned} \tag{36}$$

where k is selected such that $\phi_k^T\phi_k \geq \phi_i^T\phi_i$.

One may verify that the term S for (36) is given by $S = 2\phi_k^T\phi_k - \frac{2}{N}\sum_{i=0}^{N-1}\phi_i^T\phi_i$. The first eigenvalue of $\widetilde{D}$ (corresponding to the eigenvector $\underline{w}_0 = \frac{1}{\sqrt{N}}\underline{1}$) is $\omega_0 = 4\phi_k^T\phi_k$, and the sum of the eigenvalues of $\widetilde{D}$ (sum of elements on the main diagonal) equals to $2S$. The remaining N-1 eigenvectors are orthogonal to $\underline{1}$ and since Y^TY is positive semidefinite they are all negative. Moreover, the sum of these remaining N-1 eigenvalues is therefore: $\frac{-4}{N}\sum_{i=0}^{N-1}\phi_i^T\phi_i$.

The corresponding lower bound is (as $q \to 0$):

$$\frac{1}{2}\left(\omega_0 + \sum_{i=1}^{N-1}(1-2m_iq)\cdot\omega_{N-i}\right) - S = q\sum_{i=1}^{N-1}m_i\cdot(-\omega_{N-i}) \tag{37}$$

Since $m_i \geq 1, \quad i = 1,2,\ldots,N-1$,

$$q\sum_{i=1}^{N-1}m_i\cdot(-\omega_{N-i}) \geq q\sum_{i=1}^{N-1}(-\omega_{N-i}) = \frac{4q}{N}\sum_{i=0}^{N-1}\phi_i^T\phi_i \tag{38}$$

Since the r.h.s. of (38) is the bound in (35), the proposed lower bound is never lower (and hence better) than the one in [15].

As a synthetic example, we consider the following set of codevectors:

$$\phi_0 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad \phi_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \tag{39}$$

The asymptotic lower bound according to [15] is $D_C \geq 48q$. The distance matrix in this example:

$$D = 32\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{40}$$

Since every channel error results in the same distortion, the channel distortion is *independent* of the index assignment. The first step in calculating the lower and upper bounds is the evaluation of the *weighted distance matrix*, $\widetilde{D}$. Fortunately, the sum of elements in all rows of D is the same and therefore $\widetilde{D} = DP + PD = 0.5 \cdot D$. The Perron-Frobenius eigenvalue of D is 96 and the corresponding eigenvector is $\underline{1}$. The remaining eigenvectors of D are orthogonal to the vector $\underline{1}$ and their corresponding eigenvalue are all equal to (-32). The proposed lower and upper bounds obtained by substituting these eigenvalues into (24) coincide, resulting in $D_C = 64q$ (as compared with $D_C \geq 48q$ from [15]).

B. Simulation results

1. A 3-bit PDF optimized quantizer and a Gaussian source under the BSC

For 3-bit quantizers there are $8! = 40,320$ possible assignments so that an exhaustive search is possible. We consider a 3-bit PDF-optimized scalar quantizer [24, Ch. 4], for a Gaussian source and a BSC. The bounds obtained from (23) are shown in Fig. 2.

It can be seen that the slope of the lines is roughly 10dB/decade, i.e., reducing the bit error rate by a factor of 10 results in a 10dB lower distortion. The channel distortion is approximately proportional to the bit error rate. The upper bound is about 0.3dB higher than channel distortion due to the worst possible index assignment. The lower bound is 0.8dB lower than the distortion for the best assignment (found here by exhaustive search).

2. A 4-bit uniform quantizer and a uniform source using a (7,4) Hamming error-correcting-code under the BSC

Consider a 4-bit uniform scalar quantizer and a uniform source. The digital information is sent through a BSC utilizing a (7,4) Hamming Error Correcting Code [23]. The channel matrix H is different from the BSC case. We examine a single entry of the channel transition matrix H in this case. Assume the encoder needs to transmit the index i. The corresponding Hamming codeword $c(i)$ (7 bits) is sent through the BSC. Each Hamming-code decoder output $c(j)$ is a result of one of 8 possible BSC outputs (Hamming-code decoder input). These are the Hamming codeword $c(j)$ and its Hamming-1 distance neighbors. Each entry of H is therefore a sum of 8 probabilities:

$$\begin{aligned}\{H\}_{i,j} &= \text{Prob}\{\text{ Quantizer Index } j \text{ received} | \text{ Quantizer Index } i \text{ transmitted }\} = \\ &= \sum_{d_H(k,c(j))\leq 1} \text{Prob}\{ k \text{ received} | \text{codeword } c(i) \text{ transmitted }\}\end{aligned} \tag{41}$$

The bounds are shown in Fig. 3.

It can be seen that the slope of the graphs is 20dB/decade, i.e., reducing the bit error rate by a factor of 10 results in a 20dB lower distortion. The channel distortion is approximately proportional to the square of the bit error rate. The upper bound is about 0.5dB away from the worst random assignment (out of 10^6 index assignments) found in simulations. The proposed lower bound coincides with the performance of the NBC and it can be shown, using the same technique presented in [3], that the NBC is also optimal for this case. The ratio between the upper and lower bounds is approximately 3.6dB, compared with 6dB for the BSC without channel protection. The implementation of the channel protection brought the bounds closer together, decreasing the effect of index assignment.

3. Two-dimensional PDF-optimized vector quantizer for a Gauss-Markov source under the BSC

We examine here a set of two-dimensional vector quantizers, designed for a Gauss-Markov source, with correlation $\rho = 0.5$, and different sizes. The vector quantizers were designed using the well-known LBG algorithm [1],[24]. The digital information is sent through a BSC.

The results obtained are shown in Table 1.

			Performance ratios (dB)			
VQ index size (bits)	No. of possible assignments	Error correcting code	$\frac{\text{U. bound}}{\text{L. bound}}$	Upper bound gap	Lower bound gap	$\frac{\text{Ave. dist.}}{\text{L. bound}}$
4	$2 \cdot 10^{13}$	-	3.6	0.6	0.6	2.0
4	$2 \cdot 10^{13}$	(7,4) Hamming	2.0	0.3	0.2	0.7
6	$1 \cdot 10^{89}$	-	5.7	0.7	1.3	3.6
8	$9 \cdot 10^{506}$	-	7.2	0.9	2.0	4.8

Table 1- Bounds characteristics for two-dimensional vector quantizers designed for a Gauss-Markov source, with correlation $\rho = 0.5$.

It can be seen that the distance between upper and lower bounds increases with the complexity of the quantizer. The gap between the upper bound and the worst assignment found in simulations as well as the gap between the lower bound and the best assignment found in simulations (out of 10^6 index assignments) also expand with VQ size.

Unfortunately, due to the huge amount of possible assignments and the sub-optimality of the index-switching algorithm, we cannot state at this point if these gaps are due to an inadequate index searching mechanism or as a result of insufficient bound tightness (or both). As seen earlier, the error-correcting code brought the bounds closer together. When an error-correcting code is applied, the relatively small ratio between the average distortion and the lower bound suggests that in this case one can resort to just choosing the best of several random assignments.

4. Three-dimensional 8-bit PDF-optimized vector quantizer for quantizing images in the L*a*b* color space

The L*a*b* color space was developed by the CIE [28] in order to better match color representation to human color perception. Pixel colors are organized in three components: An a-chromatic (luminance) component L*, and two chromatic ones: a* and b*. Because color difference perception over the L*a*b* color space is approximately uniform, the Euclidean distance measure is considered to be an appropriate distance measure in this color space. The non-linear transformations between RGB and L*a*b* spaces can be found in [28]. We examine here an 8-bit, N=256, vector quantizer from [29]. The computed bounds are shown in Fig. 4.

The upper bound is about 0.6dB higher than the channel distortion due to the worst index assignment obtained in the simulations (out of 10^6 index assignments). The lower bound is 1.5dB lower than the distortion for the best assignment obtained in simulations. The ratio between the upper and lower bounds is 8.8dB, suggesting that a significant performance gain can be achieved by a good index assignment.

VI - Conclusions

In this paper we have presented upper and lower bounds (and a related expression for the average performance) of the distortion due to channel errors for vector quantizers operating under channel errors, over all possible index assignments. These bounds are based on a method known as a projection of the quadratic assignment problem. Special properties for the VQ index assignment were used to obtain bounds that are tighter than the projection bounds for the general QAP. The bounds enable the VQ designer to estimate the gain that may be obtained by a search for an efficient index assignment. Together with the average performance, the designer may evaluate the performance of a given index assignment.

Analytical and numerical examples were given for the Binary Symmetric Channel, with and without error correction. For 3 bits (8 levels) quantizers, the bounds were compared with the best and worst assignment and appear to be tight. For quantizers with 4 bits and more, the bounds were compared with "good" and "poor" assignments obtained in simulations using a sub-optimal index-switching algorithm.

For low and intermediate size vector quantizers, under the binary symmetric channel, the bounds are reasonably close to the performance of the assignments found in simulations.

For large size VQs, there exists a larger gap between the bounds and the simulation results. The huge number of possible index assignments and the sub-optimality of the index-switching algorithm leave the tightness issue of the proposed bounds to further study.

Utilization of error correction decreases the gap between the lower and the upper bounds, and both bounds are tighter. This result agrees with the intuition that channel protection reduces the importance of index assignment.

Acknowledgment

The authors would like to thank the anonymous reviewers for their insightful suggestions and helpful comments.

References

[1] Gersho A. and Gray R.M., *Vector Quantization and Signal Compression*, Kluwer Academic Publishers, Boston 1991

[2] Ben-David G. and Malah D., "On the Performance of a Vector-Quantizer under Channel Errors", Signal Processing VI Theories and applications, Proceeding of EUSIPCO-92, Elsevier Amsterdam, pp. 1685-1688, 1992.

[3] Ben-David G. and Malah D., "Properties of the Natural Binary Code Assignment for Uniform Scalar Quantizers under Channel Errors", Proc. of the ECCTD-93, Elsevier Amsterdam, pp. 773-778, 1993.

[4] Ben-David G. and Malah D., "Simple Adaptation of Vector-Quantizers to Combat Channel-Errors", Proc. of the 6th IEEE DSP Workshop, IEEE NY, pp. 41-44, 1994.

[5] Ben-David G., *Vector and Scalar Quantization under Channel Errors*, Ph.D. Thesis, Technion-IIT, Israel, June 1995 [In Hebrew]

[6] McLaughlin S.W., Neuhoff D.L., Ashley J., "The optimality of the Natural Binary Code", *Proc. of the Joint DIMACS / IEEE Workshop on Coding and Quantization*, Oct. 1992.

[7] Farvardin N. and Vaishampayan V., "Optimal Quantizer design for Noisy Channels: An Approach to Combined Source-Channel Coding ", *IEEE Trans. Inform. Theory*, pp. 155-159, Nov. 1987.

[8] Farvardin N., "A study of Vector Quantization for Noisy Channels", *IEEE Trans. Inform. Theory*, Vol. 36, No. 4, pp. 799-809, Jul. 1990.

[9] Panos M. Pardalos, Franz Rendl and Henry Wolkowicz, "The Quadratic Assignment Problem: A Survey and Recent Development", *DIMACS Series in Discrete Mathematics and Theoretical ComputerScience*, pp. 1-42, Vol. 16 1994

[10] Gersho A. and Zeger K.A., "Vector Quantizer Design for Memoryless Noisy Channels", *Proc. of IEEE Int. Conf. Comm.*, pp. 1593-1597 June 1988.

[11] Gersho A. and Zeger K.A., "Pseudo-Gray Coding", *IEEE Trans. Comm.*, Vol. 38, No., pp. 2147-2158, 12 Dec. 1990.

[12] De-Marca J.R.B., and Jayant N.S. "An algorithm for Assigning Binary Indices to the Codevectors of a multi-Dimensional Quantizer", *Proc. of IEEE Intl. Conf. Comm.*, Seattle WA, pp.1128-1132, June 1987.

[13] Knagenhjelm P., "A recursive Design Method for Robust Vector Quantization", *Proc. Intl. Conf. on Signal Processing Applications and Technology*, Boston-92, pp. 948-954, Nov. 1992.

[14] Knagenhjelm P., "How good is your assignment?", *Proc. IEEE ICASSP 93*, Minneapolis, Minnesota, pp. 423-426, Apr. 1993.

[15] Knagenhjelm P. and Agrell E., "The Hadamard transform – a tool for index assignment", *IEEE Trans. On Information Theory,* Vol. 42, no. 4, pp. 1139-1151, July 1996

[16] Skoglund M., "On Channel-Constrained Vector Quantization and Index Assignment for Discrete Memoryless Channels", *IEEE Trans. On Information Theory,* Vol. 45, no. 7, pp. 2615-2622, Nov. 1999

[17] Gilmore P.C., "Optimal and Suboptimal Algorithms for the Quadratic Assignment Problem", *J. Soc. Indust. Appl. Math*, Vol. 10, No. 2, pp. 305-313, June 1962.

[18] Hall K. "An r-dimensional quadratic assignment placement algorithm" *Management Science*, 77(3), pp. 219-229, Nov.1970.

[19] Talbot N.L.C. and Calway G.C., "A Fast Index Assignment Method for Robust Vector Quantisation of Image Data", *Proc. 1997 Int. Conf. On Image Proc. (ICIP '97)*

[20] Pan, J.-S., Shieh, C.-S., and Chu, S.-C., "Comparison study on VQ codevector index assignment", *Proc. of the Fifth International Conference on Spoken Language Processing (ICSLP'98)*, Paper no. 0031, pp. Th4Q1-Th4Q4, Sydney, Australia, 1998.

[21] Lancaster P. and Tismenetsky M., *The Theory of Matrices*, Academic-Press, Orlando, 1985.

[22] Hadley S.W., Rendl F. and Wolkowicz H., "A New Lower Bound via Projection for the Quadratic Assignment Problem", *Math. Oper. Res.*, vol. 17, no. 3, pp. 727-739, 1992

[23] Taub H. and Schilling D.L., *Principles of Communication Systems*, Second Ed., McGraw-Hill, New-York, 1986.

[24] Jayant N.S. and Noll P., *Digital Coding of Waveforms,* Prentice-Hall 1984.

[25] Crimmins T.R., Horwitz H.M., Palermo C.J and Palermo R.V., "Minimization of Mean-Square Error for Data Transmitted via Group Codes", *IEEE Trans. On Information Theory,* Vol. 15, no. 1, pp. 72-78, Jan. 1969

[26] Méhes A. and Zeger K., "Randomly Chosen Index Assignments Are Asymptotically Bad for Uniform Sources", *IEEE Trans. On Information Theory,* Vol. 45, no. 2, pp. 788-794, Mar. 1999

[27] Zeger K. and Manzella V., "Asymptotic Bounds on Optimal Noisy Channel Quantization Via Random Coding", *IEEE Trans. On Information Theory,* Vol. 40, no. 6, pp. 1926-1938, Nov. 1994

[28] Central Bureau of the Commission Internationale de L'Eclairage (CIE), *Colometry*, CIE publication No. 15.2 1986

[29] Spira A., *Image Paletization and Compression for Color Limited Displays*, M.Sc. Thesis, Technion-IIT, Israel, July 2000 [In Hebrew]

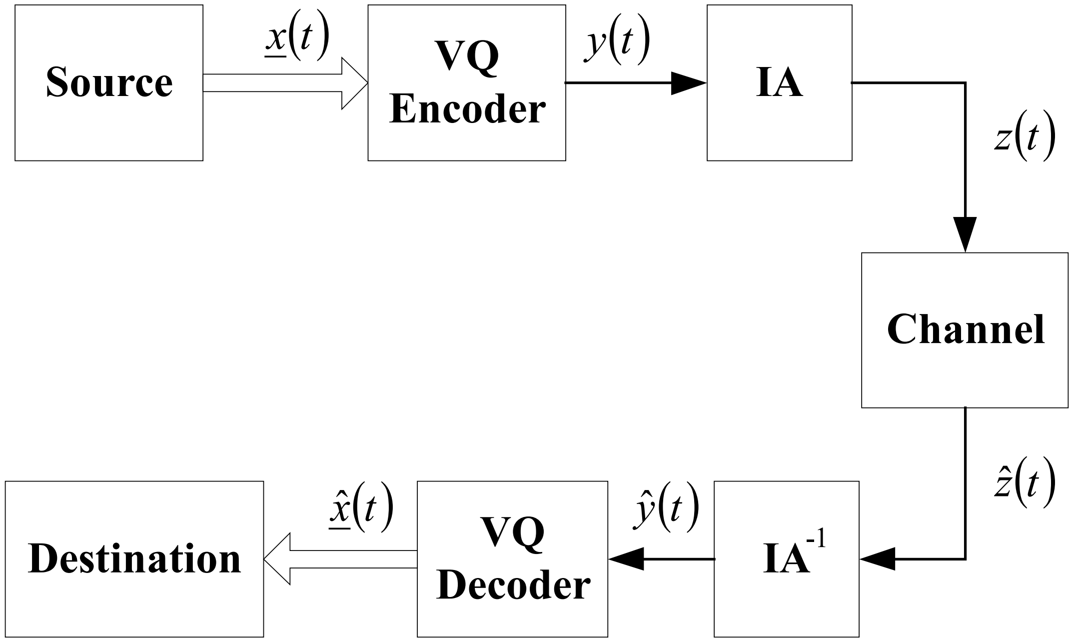

Fig.1 – Vector-Quantization based communication system

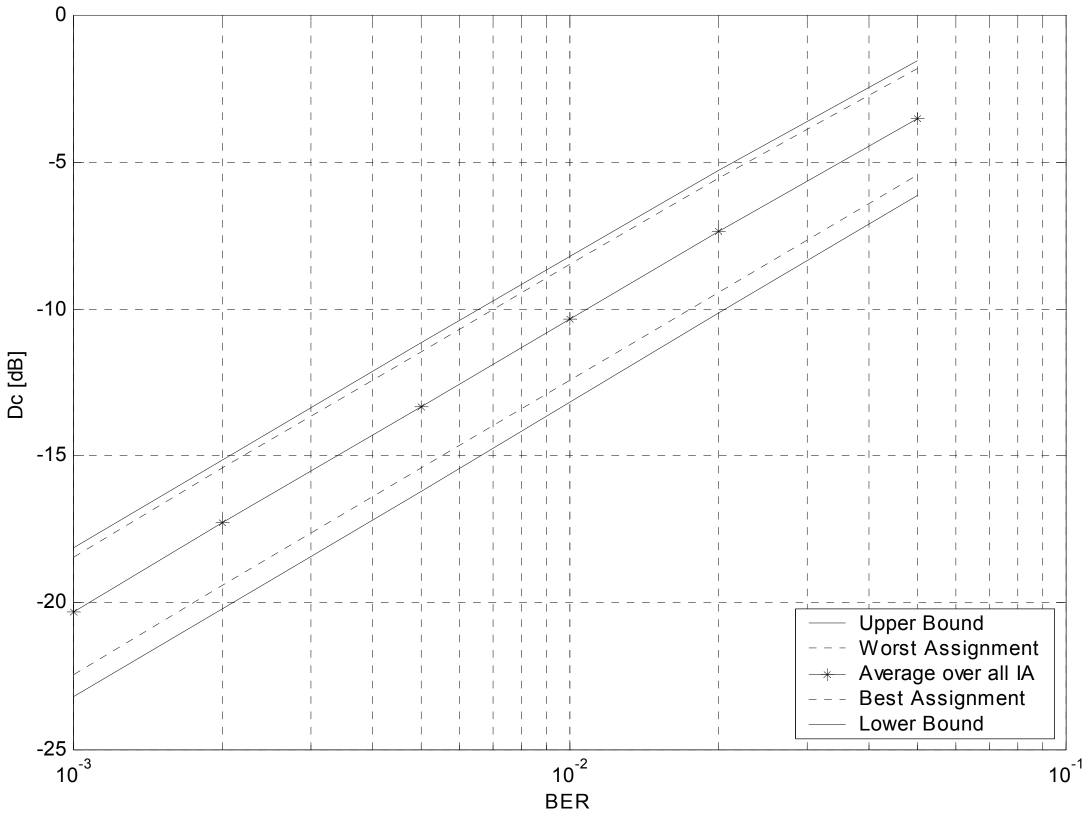

Fig. 2 – Upper and lower bounds and the average channel-distortion, over all possible index assignments, for a 3-bit PDF-Optimized Scalar Quantizer and a Gaussian source under the BSC. The bounds are compared with the best and worst assignments.

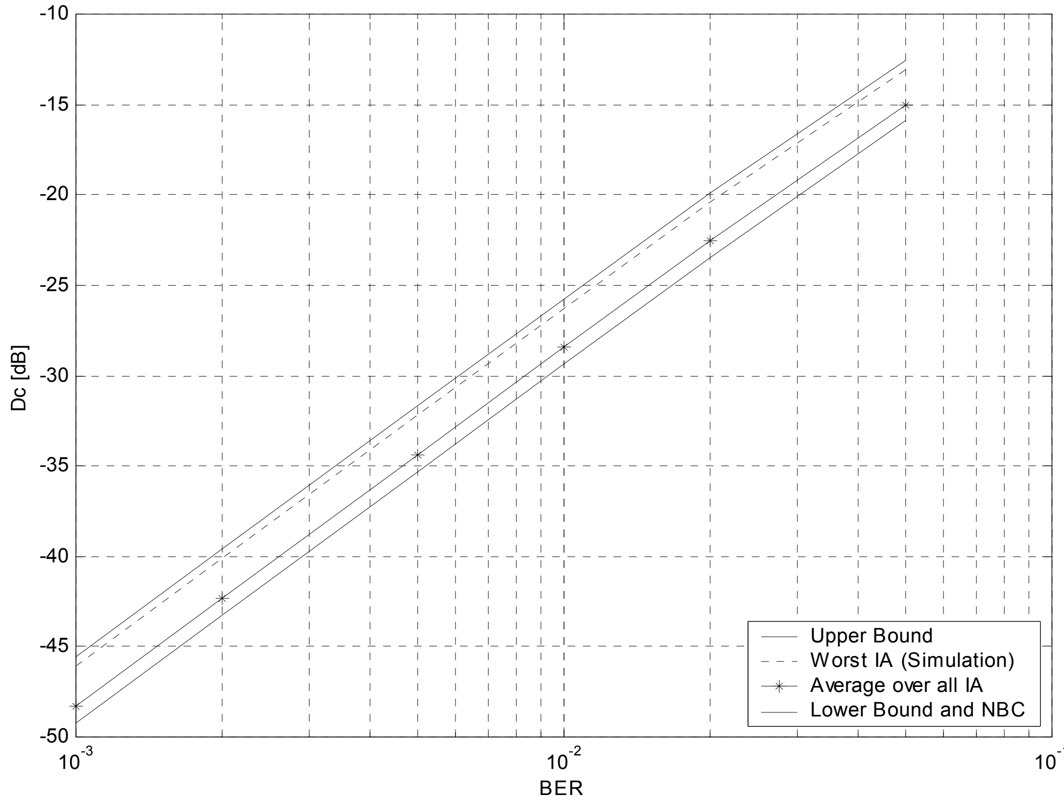

Fig. 3 – Upper and lower bounds and the average channel-distortion, over all possible index assignments, of a 4-bit Uniform Scalar Quantizer and a uniform source under the BSC with a (7,4) Hamming Error-Correctng-Code. The upper bound is compared with a "poor" assignment obtained by simulations. The lower bound coincides with the performance of the Natural Binary Code.

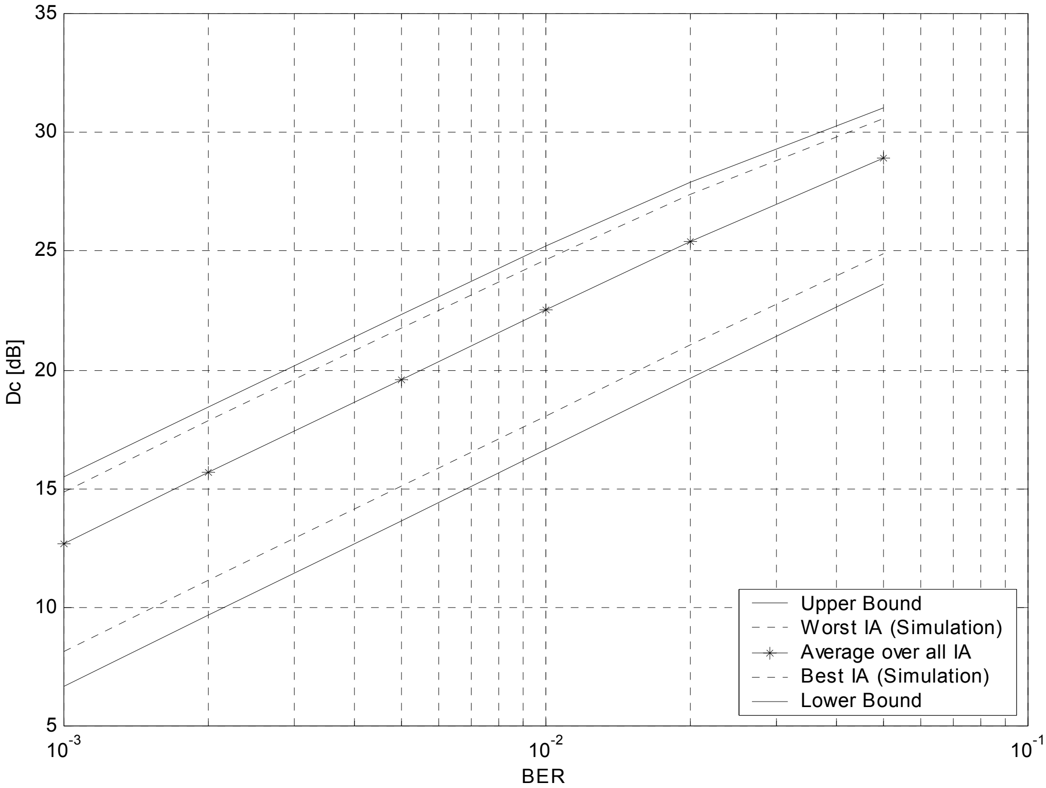

Fig. 4 – Upper and lower bounds and the average channel-distortion, over all possible index assignments, of an 8-bit L*a*b*-space Image Vector Quantizer under the BSC. The bounds are compared with "good" and "poor" assignments attained in simulations.