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On the Performance of a Vector-Quantizer under Channel Errors

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Abstract

Vector-Quantization (VQ) is an effective and widely-implemented method for low-bit-rate communication of speech and image signals. A common assumption in the analysis of VQ systems is that the compressed digital information is transmitted through a perfect channel. Under this assumption, quantizing distortion is the only factor in output signal fidelity. However, in physical channels, errors may be present and degrade overall system performance. In this paper, the effect of channel-errors on a VQ system is studied. Bounds on the distortion in general VQ schemes due to memoryless-channel errors are presented. The paper concludes with numerical results and asymptotic properties for the special case of scalar-quantizers and the Binary-Symmetric-Channel.

I. Introduction

Vector-Quantization (VQ) (or *Scalar-Quantization* as a special case) is a method for mapping signals into digital sequences. This sequences are transmitted or stored by a digital media. Motivation for using VQ is derived from information-theory [1]. It is proved that increasing the dimension of input vector results in higher compression ratios (lower bit-rates). On the other hand higher dimension causes longer input to output delay time and increased implementation complexity.

VQ can be found in waveform coding, and in particular for speech and image coding [2]-[9].

In most Signal Processing applications the *source* emits signal samples over an infinite alphabet. These samples should be sent to the *destination* with the highest fidelity possible. The VQ encodes the source output into a digital sequence that is transmitted through the *channel*. The *decoder*'s goal is to reconstruct source samples from the digital sequence. Analog sources cannot be represented perfectly by digital information, so some *distortion* must be tolerated. Throughout we assume that the VQ implements the so called *Block-Codes*, in which a fixed number of source samples is represented by a single channel symbol. The general structure of a VQ-based transmission system is shown in Figure 1.

In every channel transmission the VQ encodes a K-dimensional vector of source samples into a channel symbol $y = \varepsilon(\underline{x})$, where $\underline{x} = (x_0, x_1, \ldots, x_{K-1})^T$ is a block of samples which takes on values from Σ^K, where Σ denotes the source alphabet. The index n represents the channel time index. A channel symbol y is taken from a finite channel alphabet which, without loss of generality, is represented by indices $i = 0,1,\ldots,N-1$. The channel output $z(n)$ is a random mapping of its input. The decoder converts a channel symbol into an output reconstruction vector $\hat{\underline{x}} = \phi(z)$, that should be "close" to the input vector. The set of all reconstruction vectors (or *code vectors*) is the VQ codebook - B

$$B = \{\hat{\underline{x}} : \hat{\underline{x}} = \phi(i), i = 0,1,\ldots,N-1\} \tag{1}$$

The channel transmission rate is:

$$R = \frac{1}{K}\log_2 N \quad \text{bits/source symbol} \tag{2}$$

Next, we define a *distortion measure* $d(\underline{x}, \hat{\underline{x}})$ which assigns a cost to the encoding of an input vector by a reconstruction vector. We assume a context-free distortion-measure, which does not depend on source and reconstruction vectors at other instances. Distortion measures of practical interest are, for example, Squared Error Distortion and the Likelihood measure used in speech

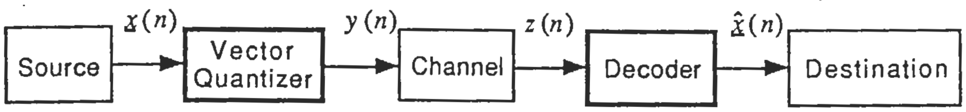

Figure 1 - General Vector-Quantization system

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coding [12].

Full knowledge of source statistics is assumed. The performance of the system is measured in terms of the average distortion:

$$D = E[d(\underline{x},\hat{\underline{x}})] \tag{3}$$

where $E[\cdot]$ denotes the expectation operator. In practice, a long-term average is used

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(\underline{x}_i,\hat{\underline{x}}_i) \tag{4}$$

In most VQ applications the channel is assumed to be noiseless, so that no errors occurs in transmission, i.e., $z(n)=y(n),\ \ \forall n$. This assumption is based upon using a channel encoder-decoder pair which realizes a perfect channel. This is possible as long as the channel capacity is not exceeded.

Upon knowledge of source statistics, Lloyd's algorithm [3] may be used to design a VQ. The design of a VQ is based in practice on a *training sequence*, which is a "long" succession of typical source outcomes. In this case no knowledge of the source statistics is assumed. The design of a VQ can be done using the LBG algorithm [10]. The LBG algorithm is important for real world applications, where a statistical probability model of the source is not available.

The paper is organized as follows. In section II we describe channel-errors and their effect on VQ-based systems. In section III a general method for obtaining tight upper and lower bounds for the average channel-error distortion is described. Numerical results for Scalar-Quantizers and the *Binary-Symmetric-Channel* (BSC) are presented in section IV.

II. Channel errors

In the discussion so far we have assumed that the channel is noiseless, i.e. channel input and output are identical z(n) = y(n), at any instant n. This may be accomplished by using Error Control Coding [1],[2]. Redundancy is introduced by a channel-encoder and used by a channel-decoder to achieve a desired error probability. In some applications channel-coding is not utilized because of complexity or bit-rate requirements. In such cases the perfect channel assumption may not be justified, and if a channel-error occurs, a wrong codevector is selected.

In [11], for example, an experiment of coding speech for Mobile Satellite communication is described. The coder examined used a pulse-excitation codebook. In order to minimize the degradation due to channel errors, the codebook indices were reorganized by iteratively switching the position of two code vectors seeking to reduce the distortion.

The effect of channel errors on specific systems can be found in [13]-[18]. Several methods for improving VQ performance under channel error are presented in [19]-[26]. The algorithms for improving the performance of VQ coding systems under channels errors are based on the following two approaches:

Redesign of the VQ cells: [19]-[22] In the presence of channel errors, and given the transmitted symbol, the received symbol is a random variable. In the process of VQ design, the distortion measure can be modified to take channel-errors into consideration so that every possible output vector is taken into account for every input vector. This modifies the partition of the space of input vectors.

Assignment of channel symbols: [23]-[28] As mentioned above, a channel error causes an incorrect decoding of a codevector. If this vector is not "far" from the codevector that should have been decoded, then a small distortion is caused. This heuristic argument motivates the search for optimal codevector indices assignment. This is a problem of a combinatorial character. For example, a 4-bit VQ has $16! \approx 2\cdot 10^{13}$ possible assignments.

III. Distortion in memoryless channels

We turn now to analyze the distortion of a VQ system due to channel-errors. In the present analysis we assume a *Discrete-Memoryless-Channel* (DMC) with the same input and output alphabet $i = 0,1,\ldots,N-1$. A DMC can be described by a channel transition matrix T whose entries are the conditional probabilities of the channel outcome given the input symbol.

$$\{T\}_{ij} = \Pr\{z(n)=j \mid y(n)=i\} \tag{5}$$

Given a distortion measure, the average distortion is:

$$D = E\{d[\underline{x},\hat{\underline{x}}]\} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\Pr\{z=j\,,\,y=i\}\, d[\phi(j),\phi(i)] \tag{6}$$

Substituting the channel transition matrix

$$\begin{aligned} D &= \sum_{i=0}^{N-1} P_i\cdot\sum_{j=0}^{N-1}\Pr\{z=j \mid y=i\}\, d[\phi(j),\phi(i)] = \\ &= \sum_{i=0}^{N-1} P_i\cdot\sum_{j=0}^{N-1} T_{ij}\, d[\phi(j),\phi(i)] = \\ &= \mathrm{trace}[P\cdot T\cdot \Delta^T] \end{aligned} \tag{7}$$

where P_i is the probability of the i-th codevector $P_i = \Pr(x_i) = \Pr[y=i]$, and $P = diag\{P_i\}$. Distortion between codevectors are the entries of $\{\Delta\}_{ij} = d[\phi(i),\phi(j)]$. Note that in (7) we assume knowledge of the codebook used by the VQ, particularly the channel-symbol assignment to every codevector.

For the special case of the *Binary-Symmetric-Channel* (BSC) the transition matrix is

$$T_{ij} = q^{\delta_{ij}}\cdot(1-q)^{L-\delta_{ij}} \tag{8}$$

where δ_{ij} is the Hamming distance between the binary representation of the integer numbers i and j, and $N = 2^L$. The parameter q is the channel Bit-Error-Rate.

In using a VQ over a noisy channel, the distortion caused by channel-error is of interest. We propose a simple technique for evaluating upper and lower bounds on the average distortion over input vector probabilities.

The bounds are found from (7) by minimizing/maximizing the average distortion by the following optimization problem:

$$\begin{aligned} &\min_{P_i}/\max_{P_i} \sum_{i=0}^{N-1} P_i \cdot \sum_{j=0}^{N-1} T_{ij}\, d[\phi(j), \phi(i)] \\ &\text{subject to:} \quad \sum_{i=0}^{N-1} P_i = 1 \\ &\qquad P_i \geq 0 \quad , \quad i = 0,1,\ldots,N-1 \end{aligned} \tag{9}$$

This is a Linear-Programming (LP) problem with one constraint. The optimal solutions (minimal and maximal) are known to belong to the finite set of *Basic-Feasible-Solutions* [27]. The k-th basic solution is

$$P_i^{(k)} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \tag{10}$$

for $k = 0,1,\ldots,N-1$. The lower and upper bounds are found by substituting (10) into (7) and selecting the minimal and maximal solutions. Note that both bounds are realized by deterministic sources.

For the BSC, asymptotic results may be obtained for small bit-error-rates. In this case the probability to obtain an error in more than one bit of the codeword index is negligible. Hence, in this case,

$$\begin{aligned} \lim_{q \to 0} \frac{D}{q} &= \sum_{i=0}^{2^L-1} P_i \cdot \sum_{j=0}^{2^L-1} q^{\delta_{ij}-1} \cdot (1-q)^{L-\delta_{ij}}\, d[\phi(j), \phi(i)] = \\ &= \sum_{i=0}^{2^L-1} P_i \cdot \sum_{j|\delta_{ij}=1} d[\phi(j), \phi(i)] \end{aligned} \tag{11}$$

For $q \to 0$ distortion is linearly proportional to q. The ratio can be bounded over P_i in a similar LP optimization argument.

For the special interesting case of a Scalar-Uniform Quantizer in the interval [-1,1], 2's Complement (2'C) code, Squared-Error distortion, and the BSC, the result is quite simple. Note that the weight of the most significant bit in 2'C code is 1. The second significant 1/2, and so on. Thus, from (11),

$$\lim_{q \to 0} \frac{D}{q} = \sum_{i=0}^{2^L-1} P_i \cdot \sum_{m=0}^{L-1} \left(\frac{1}{2^m}\right)^2 = \sum_{m=0}^{L-1} \left(\frac{1}{2^m}\right)^2 \tag{12}$$

The series in (12) converges rapidly and equals to 1.328 for a 4-bit quantizer and to 1.333 for 8-bits.

As mentioned, upper and lower bounds are realized by a deterministic source. One may add more constraints to the LP problem in order to find tighter bounds for particular cases. For example, an energy constraint can be added in the case of Scalar-Quantizer

$$\begin{aligned} &\min_{P_i}/\max_{P_i} \sum_{i=0}^{N-1} P_i \cdot \sum_{j=0}^{N-1} T_{ij}\, d[\phi(j), \phi(i)] \\ &\text{subject to:} \quad \sum_{i=0}^{N-1} P_i = 1; \quad \sum_{i=0}^{N-1} P_i \phi^2(i) \leq E \\ &\qquad P_i \geq 0 \quad , \quad i = 0,1,\ldots,N-1 \end{aligned} \tag{13}$$

where E is the energy-constraint parameter. This problem has $\binom{n}{2}$ basic solutions to evaluate.

IV. Numerical results

Upper and lower bounds on the average distortion were computed for some special cases. For the 8-bit Scalar-Uniform-Quantizer in the interval [-1,1], 2'C code, Squared-Error distortion, and the BSC, bounds are plotted in Figure 2. The linear approximation $D \approx 1.333q$ is shown by a dashed line.

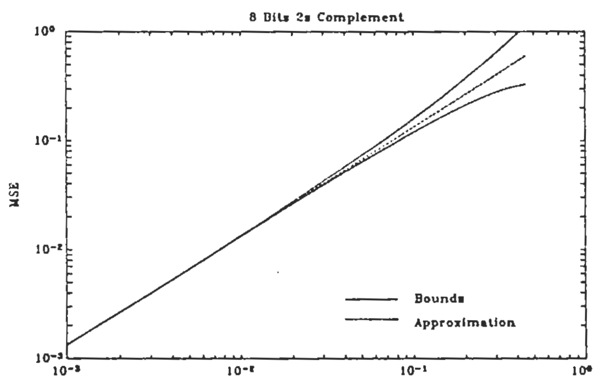

Figure 2 - Bounds on the MSE of an 8-bit linear quantizer, 2's complement code, and BSC

It is clearly seen that the bounds coincide even for a bit-error-rate as high as 0.1. The one-bit-error approximation (12) is therefore justified for those values.

For the same quantizer with Gray code representation, the bounds are not so close, as shown in Figure 3. In order to achieve tighter bounds, energy constraints were added. The lower bound was not affected by the constraints. The reason is that the lower bound is generated by the deterministic source that emits the value - 0.0. This source complies, of course, with every energy limitation. Upper bounds are plotted with E as a parameter. Similar graphs for the μ-law coder [5] are shown in Figure 4.

V. Conclusions

We have introduced a simple method to bound the distortion of a VQ-based system under channel errors. This

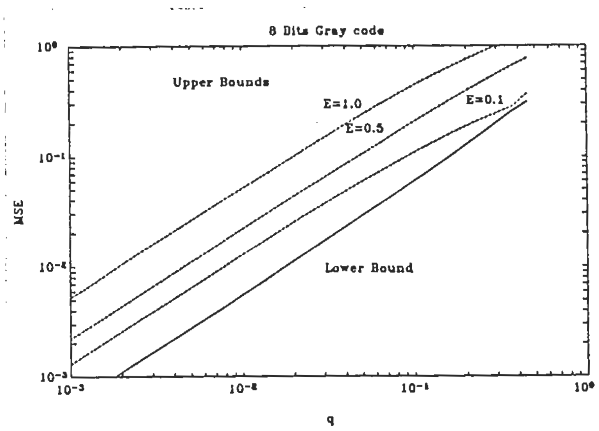

Figure 3 - Bounds on the MSE of an 8-bit linear quantizer, Gray code, and BSC

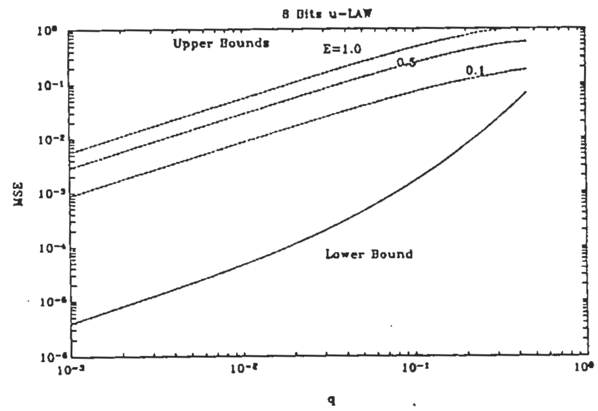

Figure 4 - Bounds on the MSE of an 8-bit, μ-law quantizer, and BSC

method involve standard LP computation. It was shown that upper and lower bounds are realized by deterministic-sources. We also presented how energy constraints can be added to obtain tighter bounds. For a scalar quantizer and the BSC, numerical and asymptotic results were shown. An interesting property of the uniform scalar quantizer with 2'C code is that the upper and lower bounds coincide for a wide range of channel bit-error-rates. For other codes the gap between the two bounds can be reduced by introducing energy constraints.

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