MORPHOLOGICAL REDUCTION OF SKELETON REDUNDANCY

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ABSTRACT

In this work we study morphological *methods to reduce the amount of redundant points in the Skeleton representation of images. The advantage of removing redundant points using* morphological operations only, *lies in the computational efficiency of these operations, when implemented on parallel machines.*

We present a generic approach to obtain redundancy-reduced Skeletons, which yields morphological closed formulae to obtain a Skeleton with less redundant points than the regular one. Although not yet able to completely remove the redundancy in the general case, the generic approach is shown to provide a redundancy-free representation, for a particular important case.

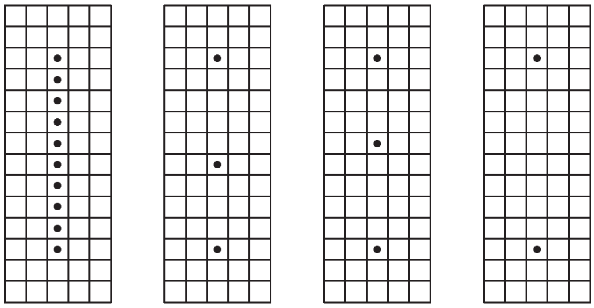
We also derive, from the generic approach, a morphological formula to obtain the Essential Points of the Skeleton, which are points that cannot be removed from the Skeleton without affecting the error-free *characteristic of the representation.*

I. INTRODUCTION

The Morphological Skeleton is a *compact* error-free representation of images, a property useful for lossless image data compression.

However, some authors have noted the fact that the Skeleton is a redundant representation, i.e., some of its points may be discarded without affecting its error-free characteristic. In some applications, such as coding, no importance is attributed to the Skeleton shape or its connectivity, but only to its *ability to fully represent images in a compact way.* In such applications, it is of interest to remove redundant Skeleton points, so that the representation contains as few as possible points.

For this purpose, Maragos and Schafer defined in [1] a *Minimal Skeleton* as being any set of points from the Skeleton which *fully represents* the original image and does not so if any of its points is removed. A Minimal Skeleton always exists since in the worst case it is the Skeleton itself. On the other hand, there can be more than one Minimal Skeleton for a given image. Fig. 1(a) shows a binary picture and its Morphological Skeleton computed with a 3x3 square structuring-element. Fig. 1(b)-(c) show two of the Minimal Skeletons of Fig. 1(a).



(a) (b) (c) (d)

Fig. 1: (a) A shape and its Skeleton (computed with a 3x3 square), (b)-(c) two of its Minimal Skeletons, (d) the Essential Points.

Maragos and Schafer propose in [1] an algorithm for finding a Minimal Skeleton from the Skeleton representation of a binary image. However, this algorithm is not fully morphological and therefore cannot be efficiently implemented on a parallel machine, in contrast to the Morphological Skeleton itself which is amenable to a parallel implementation. A fully morphological algorithm for finding Minimal Skeletons could take advantage of the parallel properties of the morphological operations and perform the computation in a more efficient way.

Sapiro and Malah defined in [2, 3] an *Essential Point* of the Skeleton as any Skeleton point that cannot be removed from the original Skeleton without affecting its error-free property. The Essential Points are contained in any Minimal Skeleton, although usually are not sufficient for exact reconstruction. The set of Essential Points is unique and it is typically the major part of Minimal Skeletons (90% and more) [2]. Because of the above properties, Sapiro and Malah suggested in [2] that the Essential Points of the Skeleton should be found first, and then the remaining Minimal Skeleton points could be searched for in a more efficient way. The Essential Points of the shape in Fig. 1(a) are shown in Fig. 1(d). Notice that they are present in the two Minimal Skeletons shown in the figure.

Another important related topic is the "Reduced Skeleton" defined by Maragos in [4]. The Reduced Skeleton has fewer representation points than the regular Skeleton and it is also error-free. It is not a Minimal Skeleton but it is obtained by morphological operations only. (The mathematical definition is reviewed in section II below).

In this work, we propose a generic morphological approach to obtain "general" Reduced Error-Free Morphological Skeletons, which gives Marago's Reduced Skeleton as a particular case. The approach also leads to a Reduced Skeleton which has less points than Maragos' Reduced Skeleton and is also error-free. We also extend the approach to *Multi-Structuring-Element Skeletons* (MSES) as well (the MSES was introduced in [5]) and show that for a particular case of MSES the approach leads to a redundancy-free representation (Minimal MSES).

We also present an algorithm (based on the same generic approach) to extract the set of Essential Points of a Skeleton and discuss a fast alternative to obtain a good approximations of it.

II. REDUCED SKELETONS

The concepts discussed in this paper are suitable for both binary images and grayscale images, but we consider here only the binary case. The images may be continuous sets (sets in R^2) or discrete sets (sets in Z^2).

A. Types of Redundant Points

Let us consider a collection of subsets $\{T_n\}$ which represents a given binary image X in the following way:

$$X = \bigcup_n T_n \oplus nB. \quad (1)$$

where $\oplus$ stands for morphological dilation, and B is a pre-defined structuring-element. The parameter n may assume all the non-negative continuous values (if X and B are continuous sets) or it may assume only discrete values $n = 0, 1, \ldots$ (for X and B which are both continuous or both discrete).

A point t belonging to the subset of order n represents an element nB translated to t:

$$\{t\} \oplus nB = nB_t \quad (2)$$

where $B_t \triangleq \{t + b \mid\ b \in B\}$.

If $t \in T_n$ is redundant, then the element it represents (nB_t) is contained in a region represented by all the other representation points, i.e.,

$$nB_t \subseteq \left(\bigcup_{m \neq n} T_m \oplus mB \right) \cup [(T_n - \{t\}) \oplus nB] \quad (3)$$

Each redundant point can be classified into one or more of the following redundancy categories:

Single-Element Redundancy: if there exists at least *one* element *bigger* than nB that covers nB_t, i.e.:

$$\exists m > n,\ \exists z \mid nB_t \subset mB_z \quad (4)$$

Future-level Redundancy: if there exists a *union* of elements bigger than nB that covers nB_t, i.e.:

$$nB_t \subseteq \bigcup_{m>n} T_m \oplus mB \quad (5)$$

Note that every "Single-Element" redundant point is also a "Future-level" redundant point.

Past-level Redundancy: if there exists a *union* of elements smaller than nB that covers nB_t, i.e.:

$$nB_t \subseteq \bigcup_{m<n} T_m \oplus mB \quad (6)$$

Interlevel Redundancy: if there exists a union of elements with size different from n that covers nB_t, i.e.:

$$nB_t \subseteq \bigcup_{m \neq n} T_m \oplus mB \quad (7)$$

Hence, every future-level or past-level redundant point is also an interlevel redundant point.

Intralevel Redundancy: if the point is not Interlevel redundant, i.e. if *every* set of elements that covers nB_t (but is not containing nB_t) contains at least one element of size n.

B. The Morphological Skeleton and Its Redundancy

The Morphological Skeleton representation of a binary image X, with a given binary structuring-element B, is a collection of sets $\{S_n\}$, which satisfies (1) for $T_n = S_n, \forall n$. The set S_n is called *Skeleton subsets of order n* and is given by:

$$S_n = X \ominus nB - (X \ominus nB) \circ (\Delta n)B \quad (8)$$

where $\Delta n = dn$ (an infinitesimally small number) if n is continuous, or $\Delta n = 1$ if n assume only natural values. The symbols $\ominus$, $\circ$ and $\bullet$ denote, respectively, binary erosion, opening and closing. The minus sign denotes here *set-difference*.

It is well known that the Skeleton, being the set of points which are centers of maximal elements, does not contain redundant points from the "Single-Element" category, i.e., it does not contain "Single-Element Redundancy". On the other hand, it may contain redundant points from all the other categories.

For demonstration, Fig. 2 shows a continuous binary image composed by the union of 2 disks, P and Q, which are centered at the points p and q, respectively. The Skeleton of the shape, computed with a continuous disk as structuring-element, for continuous values of n, is the segment $[p, q]$. In this case, all the points are redundant, except p and q. The point a in Fig. 2(a) is a "Future-level" redundant point, because the element

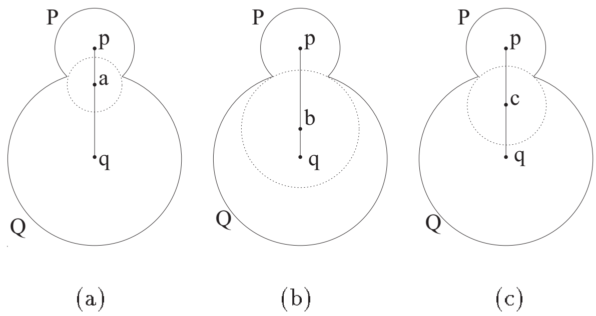


Fig. 2: Types of redundant points in the Skeleton. (a) A binary image composed by two disks (P and Q), and its Skeleton (the segment $[p,q]$). The point a is a "Future-level" redundant point, (b) the point b is a "Interlevel" redundant point, and (c) the point c is a "Intralevel" redundant point.

it represents (the dotted disk) is contained in the union of 2 bigger disks (P and Q). The point b in Fig. 2(b) is "Interlevel" redundant, because it represents a disk (the dotted one) contained in the union of a bigger disk (Q) and a smaller disk (P). The point c in Fig. 2(c) is "Intralevel" redundant, because the dotted disk, which it represents, is contained in the union of a larger disk (Q) and a disk with the same size (P), and it is not contained in any union of only larger and smaller disks.

C. The Proposed Generic Approach to Obtain Reduced Skeletons

In [1] and [2], the approach used to remove redundant points from the Skeleton was *first* to calculate the Skeleton and *then* to apply a reduction algorithm to remove the redundant points.

However, we note that the skeletonization itself is a *partial reduction process*, and we demonstrate this with the following considerations. If the Skeleton subsets S_n would have been defined as $S_n = X \ominus nB$, $\forall n$, then the exact reconstruction property (1) for $T_n = S_n$ would still be satisfied, but this "Skeleton" would contain too many points. In fact, S_0 itself would then be equal to X. Instead, the sets $[X \ominus nB] \circ (\Delta n)B$ of redundant points are removed from $X \ominus nB$ for all n in the definition of the Skeleton (8), so that a compact representation is obtained. However, as mentioned before, only the "Single-Element Redundancy" is removed this way.

We propose to remove as many redundant points as possible *during* the skeletonization process, which is fully morphological, so that a more efficient error-free decomposition than the regular Skeleton is obtained by morphological operations only.

The following relation illustrates the approach:

$$RS_n = \begin{pmatrix}\text{representation} \\ \text{points of order } n\end{pmatrix} - \begin{pmatrix}\text{redundant} \\ \text{points of} \\ \text{order } n\end{pmatrix} \quad (9)$$

where $\{RS_n\}$ are the *Reduced Skeleton subsets.*

When the representation points are the centers of elements nB, the above relation can be written as follows:

$$RS_n = \begin{pmatrix}\text{representa-} \\ \text{tion region} \\ \text{of order } n\end{pmatrix} \ominus nB - \begin{pmatrix}\text{redundant} \\ \text{region of} \\ \text{order } n\end{pmatrix} \ominus nB \quad (10)$$

Usually, the "representation region of order n" is $X \circ nB$. By replacing the field "redundant region of order n" in (10) by appropriate sets, one can obtain different Reduced Skeletons.

It can be shown that if we choose $X \circ (n + \Delta n)B$ to be the "redundant region", then we obtain a Reduced Skeleton with *no "Future-level Redundancy"*. This is because $X \circ (n + \Delta n)B$ is the region represented by the *union* of all the *maximal elements* with size greater than n:

$$X \circ (n + \Delta n)B = \bigcup_{m>n} S_m \oplus mB. \quad (11)$$

The resulting Reduced Skeleton subsets $RS_n^{(1)}$ are given by:

$$RS_n^{(1)} \triangleq (X \circ nB) \ominus nB - [X \circ (n + \Delta n)B] \ominus nB \quad (12)$$

After some simple manipulations on (12), we obtain:

$$RS_n^{(1)} = X \ominus nB - [(X \ominus nB) \circ (\Delta n)B] \bullet nB \quad (13)$$

which is the Reduced Skeleton proposed by Maragos in [4]. (In [4], $\Delta n = 1$, since only discrete values of n were considered).

Fig. 3(a) shows the result of the calculation of $RS^{(1)}$ for the binary image shown in Fig. 2. It contains the point p and the segment $[c,q]$, where c is the same "Intralevel" redundant point shown in Fig. 2(c). The points from the segment (p,c), which are "Future-level" redundant in the Skeleton, are not present in $RS^{(1)}$.

If we include in "redundant region" of equation (10) the union of all the representation elements with order smaller than n as well, we obtain an error-free Reduced Skeleton, which we denote as $RS^{(2)}$, with *no interlevel redundancy.* The union of the representation elements

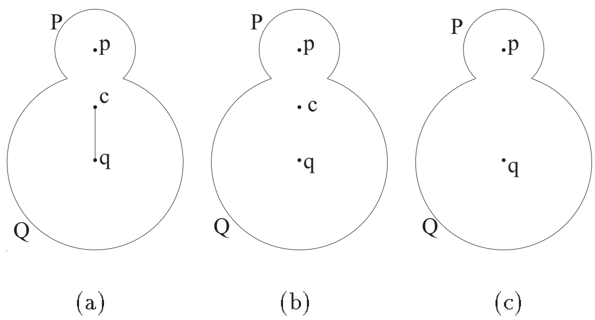


Fig. 3: (a) The same binary image as shown in Fig.2, and its Reduced Skeleton $RS^{(1)}$, (b) its Reduced Skeleton $RS^{(2)}$, (c) its only Minimal Skeleton.

with order smaller than n, which we denote as P_n, may be obtained recursively, as follows:

$$\begin{array}{l} P_{n+\Delta n} = P_n \cup (RS_n^{(2)} \oplus nB), \quad n = 0, 1, \ldots \\ P_0 = \emptyset \end{array} \tag{14}$$

Defining:

$$F_n \triangleq X \circ (n + \Delta n)B, \tag{15}$$

we may write the subsets of $RS^{(2)}$ as:

$$RS_n^{(2)} \triangleq X \ominus nB - (P_n \cup F_n) \ominus nB \tag{16}$$

Fig. 3(b) shows the result of the calculation of $RS^{(2)}$ for the same binary image as before. It contains only the points p, q and c. The points from the segment (c, q), which are "Interlevel" redundant in the Skeleton, are not present in $RS^{(2)}$. The point c, which is "Intralevel" redundant, is still present.

To obtain a Minimal Skeleton, the intralevel redundancy should also be removed. Unfortunately, we still don't know how to define a "redundant region" that would remove this kind of redundancy without affecting the property of exact reconstruction of the Reduced Skeletons. In the example of Fig. 3, the Minimal Skeleton (which is unique in this example) is shown in Fig. 3(c).

D. Extension to Multi-Structuring-Element Skeletons (MSES)

In [5], we define and discuss some of the applications of the Multi-Structuring-Element Skeleton (MSES). The MSES is a generalization of the original Skeleton, using several structuring-elements, instead of just one, representing images in a more descriptive way than the original Skeleton [5].

For the simplest case of just 2 structuring-elements, B_1 and B_2, and parameters assuming discrete values only, the MSES's subsets $S_{n,m}$ are then defined by:

$$S_{n,m} = X \ominus A(n,m) - \bigcup_{i=1}^{2} [X \ominus A(n,m)] \circ B_i \tag{17}$$

where $\{A(n,m)\}$ is a 2-parameter family of shapes generated from B_1 and B_2 in the following way:

$$A(n,m) = nB_1 \oplus mB_2 \tag{18}$$

The MSES is the center of maximal elements taken from the 2-parameter family $\{A(n,m)\}$.

Reduced MSES's may be obtained from the same generic relation (9). With the addition of the new parameter, (10) should be written as:

$$\begin{array}{rl} RS_{n,m} = & \begin{pmatrix}\text{representation region} \\ \text{of order } n,m\end{pmatrix} \ominus A(n,m) \\ & - \begin{pmatrix}\text{redundant region} \\ \text{of order } n,m\end{pmatrix} \ominus A(n,m) \end{array} \tag{19}$$

A Reduced MSES with only *intralevel* redundancy (analog to (16)) would have "representation region of order (n,m)" equals to $X \circ A(n,m)$, and "redundant region of order (n,m)" equals to the union of all the maximal elements different from $A(n,m)$.

In order to define such "redundant region", we use the *lexicographic* relation of order ($<$) in N^2:

$$\begin{array}{c} (a,b) < (c,d), \quad a,b,c,d \in N \\ \Updownarrow \\ (a < c) \text{ or } (a = c \text{ and } b < d) \end{array} \tag{20}$$

The process of the calculation of the Reduced MSES is done obeying the lexicographic order, in such a way that a subset $RS_{n,m}$ is not computed until all the subsets with index *smaller* then (n,m) are computed. After the set $RS_{n,m}$ is computed, the next subset to be computed will be $RS_{n,m+1}$, if $X \ominus A(n, m+1) \neq \emptyset$, or $RS_{n+1,0}$ otherwise.

We then define $P_{n,m}$ and $F_{n,m}$, which are analog to P_n and F_n defined in the last subsection, by the union of the representation elements *smaller* than $A(n,m)$, and the union of the maximal elements *bigger* than $A(n,m)$, respectively.

$P_{n,m}$ are found recursively in the same way as P_n, "accumulating" the representation regions $S_{n,m} \oplus A(n,m)$ for each step (n,m).

It can be shown that $F_{n,m}$ as defined above, can be computed at each step (n,m) by the formula:

$$F_{n,m} = [X \circ A(n, m+1)] \cup [X \circ A(n+1, 0)] \tag{21}$$

The proposed Reduced MSES, with no interlevel redundancy, is hence defined as:

$$RS_{n,m} \triangleq X \ominus A(n,m) - (P_{n,m} \cup F_{n,m}) \ominus A(n,m). \tag{22}$$

E. Minimal MSES

For a particular but important choice of the structuring-elements of the MSES with 2 structuring-elements, formula (22) yields a representation with *no redundant points*, i.e., a Minimal MSES. For discrete shapes, the structuring-elements which provide this result are elements containing exactly 2 points, which we call *discrete elementary directional structuring-elements.* Fig. 4 shows some examples of discrete elementary directional structuring-elements.

It is well known [1] that the ordinary Skeleton, computed with any directional structuring-element, contains no redundancy.

As an extension to this property, the Reduced MSES obtained in (22), computed with any pair of structuring-elements from Fig. 4 (or any other pair of elementary directional elements), contains no *intralevel* redundancy.

Since the Reduced MSES from (22) has no *interlevel* redundancy, the conclusion is that it contains no redundant points at all. It is therefore a Minimal MSES.

In contrast to 1-parameter families of directional shapes, in which there is little interest as kernels, the families of shapes generated by pairs of elementary

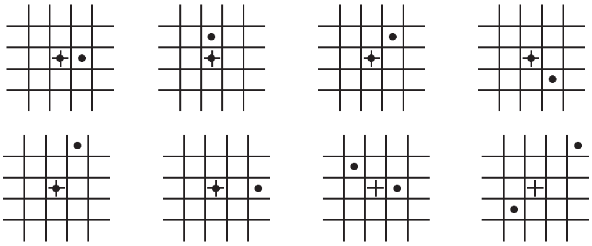


Fig. 4: Some discrete elementary directional elements: each element is composed of exactly 2 points. (The symbol "+" represents the origin).

directional structuring-elements are important ones. E.g., in the case of the *horizontal* and the *vertical* elementary structuring-elements (the first two elements shown in Fig. 4), the family $A(n,m)$ obtained is composed of all discrete rectangles.

III. EXTRACTION OF ESSENTIAL POINTS

An *Essential point* of a Skeleton Representation is defined to be a point of the Skeleton which, if it is removed from the original Skeleton, makes the exact reconstruction impossible [2, 3]. More specifically, a point t belonging to the Skeleton subset S_k is a *Essential Point of order k* iff:

$$\left(\bigcup_{n\neq k} S_n \oplus nB\right) \cup [(S_k - \{t\}) \oplus kB] \neq X \qquad (23)$$

As an example, Fig. 5(a) shows a binary image, Fig 5(b) shows its Skeleton (computed with a 3x3 square as structuring-element), Fig. 5(c) shows a Minimal Skeleton and Fig. 5(d) shows its Essential Points (which is a subset of the Minimal Skeleton).

The same approach that yields the Reduced Skeletons of the last section, also permits us to obtain the Essential Points of the Skeleton using morphological operations only. The calculation is performed at each step of the skeletonization process, so that the Essential Points of order n are obtained *before* the calculation of the Skeleton subsets of orders *greater* than n.

To extract the Essential Points, equation (9) is written in the following way:

$$EP_n = S_n - \begin{pmatrix}\text{Non-}\\ \text{Essential}\\ \text{Points of}\\ \text{order } n\end{pmatrix} \qquad (24)$$

where EP_n is the set of Essential Points of order n, and "Non-Essential Points of order n" are those Skeleton points of order n which are not Essential Points.

Equation (24) may also be written as follows:

$$EP_n = S_n - (\tilde{P}_n \cup F_n \cup R_n) \ominus nB \qquad (25)$$

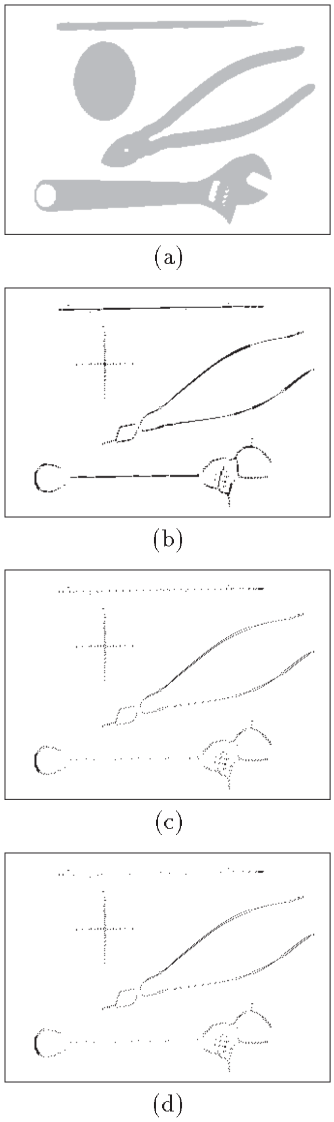


Fig. 5: (a) The binary picture "Tools", (b) its Skeleton computed with a 3×3 square, (c) a Minimal Skeleton subset of (b), (d) the Essential Points of (b).

The union $\tilde{P}_n \cup F_n \cup R_n$ refers to the region represented by all the Non-Essential Points of order n, i.e., a subregion from the representation region of order n, which is *represented more than once.*

As before, $\tilde{P}_n$ and F_n are related, respectively, to elements "smaller" and "bigger" than nB. F_n is the same as computed in (15). $\tilde{P}_n$ is computed recursively as was done for P_n in (14), "accumulating" the regions covered by Skeleton points of order smaller than n:

$$\begin{array}{l}\tilde{P}_{n+\Delta n} = \tilde{P}_n \cup (S_n \oplus nB), \quad n = 0,1,\ldots\\ \tilde{P}_0 = \emptyset\end{array} \qquad (26)$$

The sets $\{R_n\}$ are those regions which are covered *more than once by elements of size n only.* An exact expression for computing R_n is:

$$R_n = \bigcap_{s \in S_n} [(S_n - \{s\}) \oplus nB] \quad (27)$$

The outline of the proof is presented in the Appendix.

Formula (27) is an efficient way to calculate R_n only for large values of n, because in that case S_n contains only few points. For small values of n, though, there are many points in the corresponding Skeleton subsets, in which case this formula looses its efficiency.

It can be shown that formula (27) is equivalent to:

$$R_n = \bigcap_{b \in nB} [S_n \oplus (nB - \{b\})] \quad (28)$$

For small values of n, equation (28) is preferable to (27) because nB in this case contains a small number of points.

Once R_n, F_n and $\tilde{P}_n$ are found, the Essential Points of order n can be obtained by (25). Since the above sets can be obtained with morphological operations only, as shown in (15), (26), and (27) or (28), and since (25) is also morphological, the conclusion is that the extraction of Essential Points can be implemented by a morphological machine.

As we have done in section II.D, the result expressed by (25) can be adapted for MSES's, as well.

IV. CONCLUSION

A *morphological* approach for obtaining redundancy-reduced Skeletons was presented. This approach is able to remove all the *interlevel* redundancy from the Skeleton, leaving only *intralevel* redundant points. For *Multi-Structuring-Element Skeletons*, and a particular but an important choice of the structuring-elements, the approach provides a redundancy-free representation. A fully morphological method to compute a redundancy-free Skeleton (Minimal Skeleton) in the general case, both for MSES's and for the regular Skeleton, is still being sought.

A *morphological* formula for extracting the Essential Points of the Skeleton representation is also proposed.

APPENDIX

The outline of the proof of relation (27), which gives a formula to obtain the sets $\{R_n\}$, is as follows:

- The region represented by Skeleton points of order n is:

$$S_n \oplus nB. \quad (A.1)$$

- The region represented by all the Skeleton points of order n, *except* a Skeleton point $s \in S_n$ is:

$$(S_n - \{s\}) \oplus nB. \quad (A.2)$$

- The region represented *only by the point* s is the difference of the above sets:

$$S_n \oplus nB - (S_n - \{s\}) \oplus nB. \quad (A.3)$$

- The union of the above sets, for all the Skeleton points $s \in S_n$ is:

$$\bigcup_{s \in S_n} [S_n \oplus nB - (S_n - \{s\}) \oplus nB] \triangleq Y. \quad (A.4)$$

  It gives the union of those regions which are each represented by only one point of order n.

- $S_n \oplus nB - Y$ is the region represented by more than one point of order n, i.e.,

$$S_n \oplus nB - Y = R_n. \quad (A.5)$$

- Y may also be written as:

$$Y = S_n \oplus nB - \bigcap_{s \in S_n} (S_n - \{s\}) \oplus nB. \quad (A.6)$$

- relation (27) is then obtained considering the last two items, and the fact that Y is contained in $S_n \oplus nB$.

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