TRANSMISSION MATRIX OF A CLASS OF DISCRETE-TIME SYSTEMS

Indexing terms: Discrete-time systems, Time-varying systems, Linear systems

Some new properties of the transmission matrix of the class of scalar, linear, discrete-time-varying systems, of order m, which are totally m-controllable and totally m-observable, are found. The results can be applied to the design, in the time domain, of discrete-time-varying or time-invariant systems.

Introduction: The input–output relations of a single-input single-output nonanticipatory linear discrete-time-varying system, which is initially relaxed, are given by

$$y(n) = \sum_{k=n_0}^{n} h(n, k)\, u(k) \quad (n \geqslant k \geqslant n_0) \qquad (1)$$

where $u(n)$ and $y(n)$ are the input and output signals, respectively, and $h(n, k)$ is the response of the system to a unit impulse δ_{nk} (Kronecker delta), satisfying $h(n, k) = 0$ for $k > n$. For convenience, we assume that $n_0 = 0$.

The convolution summation in eqn. 1 can also be expressed in matrix form as

$$\hat{\boldsymbol{y}} = \boldsymbol{H}\hat{\boldsymbol{u}} \qquad (2)$$

where* $\hat{\boldsymbol{u}} = [u(0)\, u(1)\, u(2)\ldots]'$, $\hat{\boldsymbol{y}} = [y(0)\, y(1)\, y(2)\ldots]'$, and $\boldsymbol{H}$ is the lower triangular matrix

$$\boldsymbol{H} = \begin{bmatrix} h(0,0) & 0 & 0 & 0 \ldots \\ h(1,0) & h(1,1) & 0 & 0 \ldots \\ h(2,0) & h(2,1) & h(2,2) & 0 \ldots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (3)$$

The matrix $\boldsymbol{H}$ in eqn. 3 is known as the transmission matrix[1-3] and has been used for the analysis and design in the time domain of time-varying and time-invariant sampled-data systems.[1,4,5]

Input–output representation: Consider the system S_0, of order m, which is described by the state equations

$$\left.\begin{aligned} \boldsymbol{x}(n+1) &= \boldsymbol{A}(n)\,\boldsymbol{x}(n) + \boldsymbol{b}(n)\,u(n) \\ y(n) &= \boldsymbol{c}'(n)\,\boldsymbol{x}(n) + d(n)\,u(n) \end{aligned}\right\} \qquad (4)$$

where

$$\boldsymbol{A}(n) = \boldsymbol{A}_0(n) \triangleq \begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 & . \\ . & . & . & . \\ . & . & . & 1 \\ -a_m(n) & -a_{m-1}(n) & \ldots & -a_1(n) \end{bmatrix} \qquad (5a)$$

$$\boldsymbol{b}(n) = [b_1(n)\, b_2(n) \ldots b_m(n)]' \qquad (5b)$$

$$\boldsymbol{c}'(n) = \boldsymbol{c}'_0 \triangleq [1 \quad 0 \ \ldots \ 0] \qquad (5c)$$

As it is well known, the above canonical form of S_0 is directly related to the input–output representation of S_0, i.e. to the scalar difference equation, of order m,

$$L(n)\, y(n) = M(n)\, u(n) \qquad (6)$$

where $L(n)$ and $M(n)$ are operators given by

$$L(n) = \sum_{i=0}^{m} a_i(n)\, E^{m-i} \quad (a_0 = 1) \qquad (7a)$$

$$M(n) = \sum_{i=0}^{m} \beta_i(n)\, E^{m-i} \quad \{\beta_0(n) = d(n+m)\} \qquad (7b)$$

in which E is the advance operator, i.e. $Ey(n) = y(n+1)$.

The coefficients $\beta_i(n)$ can easily be determined by deriving the form of eqn. 6 from eqns. 4 and 5.

The impulse-response function $h(n, k)$ of a system which is realisable in the form of eqn. 4 can be shown to be necessarily separable in n and k (as in the continuous-time case), i.e. of the form

$$h(n, k) = \sum_{i=1}^{m} f_i(n)\, g_i(k) = \boldsymbol{f}'(n)\,\boldsymbol{g}(k) \quad (n > k \geqslant 0) \qquad (8)$$

For $k = n$, however, $h(n, n) = d(n)$, as is evident from eqn. 4.

If it is further possible to describe the system in the form

* In the following, a prime denotes 'transpose'

of eqn. 6, the functions $f_i(n)$, $i = 1, 2, ..., m$, appearing in eqn. 8, are[6] linearly independent solutions of the homogeneous equation $L(n)\,y(n) = 0$. Equivalently, as can be verified, a fundamental matrix* $X_0(n)$ of S_0 is given by the Casorati Matrix[6] $C_a(n)$ defined below.

$$X_0(n) = C_a(n)$$

$$\triangleq \begin{bmatrix} f_1(n) & f_2(n) & \dots & f_m(n) \\ f_1(n+1) & f_2(n+1) & \dots & f_m(n+1) \\ \vdots & \vdots & & \vdots \\ f_1(n+m-1) & f_2(n+m-1) & \dots & f_m(n+m-1) \end{bmatrix} \qquad (9)$$

If a given impulse-response function $h(n, k)$ is realisable by a system of the form of eqn. 4, the triplet $[\boldsymbol{A}(n), \boldsymbol{b}(n), \boldsymbol{c}'(n)]$ is called a realisation. Clearly, $[\boldsymbol{I}, \boldsymbol{g}(n), \boldsymbol{f}'(n)]$ is a realisation of eqn. 8 and will be named here a *basic realisation*.

Application of equivalence transformations* to the basic realisation yields other realisations of the same $h(n, k)$. Specifically, if the Casorati matrix $C_a(n)$ is used for such a transformation (provided that it is nonsingular), one obtains

$$\boldsymbol{A}(n) = \boldsymbol{C}_a(n+1)\,\boldsymbol{C}_a^{-1}(n) = \boldsymbol{A}_0(n) \qquad (10a)$$

$$\boldsymbol{b}(n) = \boldsymbol{C}_a(n+1)\,\boldsymbol{g}(n) = [h(n+1, n) \dots h(n+m, n)]' \qquad (10b)$$

$$\boldsymbol{c}'(n) = \boldsymbol{f}'(n)\,\boldsymbol{C}_a^{-1}(n) = [1 \quad 0 \; \dots \; 0] = \boldsymbol{c}'_0 \qquad (10c)$$

Clearly, from eqns. 9 and 10, a necessary and sufficient condition for the realisability of a specified $h(n, k)$ by a system of the form of S_0 is the nonsingularity of $C_a(n)$. Furthermore, if one applies the observability criterion of Reference 7 to the basic realisation, one finds that $C_a(n)$ is an observability matrix of it. Hence, if $C_a(n)$ is nonsingular, any realisation of order m of $h(n, k)$ is totally m-observable.‡

Use of a controllability criterion[7] for the basic realisation yields the controllability matrix

$$\boldsymbol{\Gamma}_c(n, m) = [\boldsymbol{g}(n), \boldsymbol{g}(n+1), \dots, \boldsymbol{g}(n+m-1)] \qquad (11)$$

and hence S_0 is not necessarily totally m-controllable.†

Transmission-matrix properties: The class of systems which are totally m-controllable and totally m-observable is now considered. Such systems can be transformed to the form of S_0 by an equivalence transformation of the form

$$\boldsymbol{T}(n) = \boldsymbol{C}_a(n)\,\boldsymbol{X}^{-1}(n) \qquad (12)$$

where $\boldsymbol{X}(n)$ is the fundamental matrix of the given system S, and thus $\boldsymbol{X}^{-1}(n)$ transforms it to the basic realisation, whereas $C_a(n)$ transforms the basic realisation to S_0. Clearly, total m-observability is necessary and sufficient for the existence of a nonsingular $\boldsymbol{T}(n)$ (assuming $\boldsymbol{A}(n)$, and hence $\boldsymbol{X}(n)$, to be nonsingular).

Theorem 1: Let $\tilde{\boldsymbol{H}}$ be the transmission matrix of a system $\tilde{S}$, of order m, which is totally m-controllable and totally m-observable. Then, each $m \times m$ submatrix $\tilde{\boldsymbol{H}}_m(n, k^*)$ of $\tilde{\boldsymbol{H}}$, as in eqn. 13 below, is a fundamental matrix of S_0 at instant n.

$$\tilde{\boldsymbol{H}}_m(n, k^*) = \begin{bmatrix} \tilde{h}(n, k^*) & \tilde{h}(n, k^*+1) & & \tilde{h}(n, k^*+m-1) \\ \tilde{h}(n+1, k^*) & \tilde{h}(n+1, k^*+1) & & \tilde{h}(n+1, k^*+m-1) \\ \vdots & \vdots & & \vdots \\ \tilde{h}(n+m-1, k^*) & \tilde{h}(n+m-1, k^*+1) & \dots & \tilde{h}(n+m-1, k^*+m-1) \end{bmatrix} \qquad (13)$$

where k^* is an integer satisfying $0 < k^* < (n-m+1)$, i.e. $\tilde{\boldsymbol{H}}_m(n, k^*)$ is an $m \times m$ submatrix of $\tilde{\boldsymbol{H}}$ below the main diagonal.

Proof: Since $h(n, k)$ is necessarily of the form of eqn. 8, it is noted that

$$\tilde{\boldsymbol{H}}_m(n, k^*) = \boldsymbol{C}_a(n)[\boldsymbol{g}(k^*), \boldsymbol{g}(k^*+1), \dots, \boldsymbol{g}(k^*+m-1)] \qquad (14)$$

and hence from eqn. 11

$$\tilde{\boldsymbol{H}}_m(n, k^*) = \boldsymbol{C}_a(n)\,\boldsymbol{\Gamma}_c(k^*, m) \qquad (15)$$

By hypothesis, $\boldsymbol{\Gamma}_c(k, m)$ is nonsingular for all k, and hence for $k = k^*$ it is a constant nonsingular matrix. Thus, since $C_a(n)$ is a fundamental matrix of S_0, so is $\tilde{\boldsymbol{H}}_m(n, k^*)$. Q.E.D.

Corollary 1: Every $m \times m$ determinant below the main diagonal of the transmission matrix $\tilde{\boldsymbol{H}}$ of S is nonzero.

From eqns. 9 and 15, and the property that a linear combination of the m linearly independent solutions of $L(n)\,y(n) = 0$ is also a solution, we obtain

Corollary 2: Every column below the main diagonal of $\tilde{\boldsymbol{H}}$ is a solution of the homogeneous equation $L(n)\,y(n) = 0$, where $L(n)$ is the difference operator of the input–output representation of S_0.

Corollary 3: All m adjacent columns below the main diagonal of $\tilde{\boldsymbol{H}}$ are linearly independent solutions of $L(n)\,y(n) = 0$.

Corollary 4: Every $\mu \times \mu$ determinant below the main diagonal of $\tilde{\boldsymbol{H}}$, for which $\mu > m$, vanishes.

From the above results one could, theoretically, determine the order of a system from its transmission matrix if it is totally m-controllable and totally m-observable. In practice, however, because of finite accuracy in calculations and measurement errors (if $\tilde{\boldsymbol{H}}$ is found by measurements), $\mu \times \mu$ determinants for $\mu > m$ will not necessarily vanish. Yet, if one assumes a certain order m, one can obtain a representation of the form of eqn. 6 from the given $\tilde{\boldsymbol{H}}$ by applying Theorem 1 and using $\tilde{\boldsymbol{H}}_m(n, k)$ in eqns. 10 in place of $C_a(n)$. Note that, from eqn. 10b, $b_i(n)$ is given directly by the ith diagonal below the main one.

The matrix inversion required in eqn. 10a can be avoided by evaluating the determinant of eqn. 16 below and by using the relation between eqns. 5a and 7a.

$$L(n)\,y(n) = \frac{1}{\Delta^{11}} \begin{vmatrix} y(n) & \tilde{h}(n, k^*) \; \dots \; \tilde{h}(n, k^*+m-1) \\ y(n+1) & \\ \vdots & \tilde{\boldsymbol{H}}_m(n+1, k^*) \\ y(n+m) & \end{vmatrix} \qquad (16)$$

where $0 < k^* < (n-m+1)$ and $\Delta^{11} = |\tilde{\boldsymbol{H}}_m(n+1, k^*)|$.

Note that eqn. 16 does satisfy (theoretically) $L(n)\,f_i(n) = 0$, $i = 1, 2, \dots, m$, as required.

Conclusions: It has been shown that proper submatrices of the transmission matrix of a system which is totally m-controllable and totally m-observable are fundamental matrices of the canonical representation given by eqns. 4 and 5. Other related theoretical properties are found as well. The results apply to both time-varying and time-invariant systems and can be utilised in the design or identification of discrete-time systems.

This work was supported in part by the National Research Council of Canada under grant A-8149.

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14th October 1971

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* Here, a matrix $X(n)$ is called a fundamental matrix of eqn. 4 if it satisfies[8] $X(n+1) = A(n)\,X(n)$. This matrix is not unique since $X(n)\,K$, where K is a constant nonsingular matrix, is also a fundamental matrix of eqn. 4. For the particular case where $K = X^{-1}(n_0)$, the unique matrix $\Phi(n, n_0) = X(n)\,X^{-1}(n_0)$ (known also as a 'transition matrix') satisfies $\Phi(n_0, n_0) = I$

† That is, transforming the state vector by a linear transformation $z(n) = T(n)\,X(n)$, where $T(n)$ is a nonsingular matrix

‡ A system is said here to be totally m-observable if it is completely observable[7] on each subsequence $[n_1, \dots, n_2]$ such that $n_2 - n_1 = m$, a fixed integer. It is said to be totally m-controllable if it is completely controllable[7] on each subsequence as above