SYNTHESIS OF LINEAR DISCRETE-TIME-VARYING SYSTEMS

Indexing terms: Discrete-time systems, Linear systems, Step response, Control-system synthesis

The synthesis of a linear discrete-time-varying system from its specified impulse-response matrix $\boldsymbol{H}(n, k)$ is considered. The results include a direct extension of those obtained for the continuous-time counterpart problem, and a simple decomposition method of $\boldsymbol{H}(n, k)$ which is readily extendable to the continuous-time case.

Introduction: A notable advantage of replacing a continuous-time-varying system by a discrete equivalent one is the simplicity with which the latter can be realised, because the time-varying properties are obtained just by varying the multiplying coefficients with time. If the impulse-response matrix $\boldsymbol{H}(t, \tau)$ of the linear continuous-time-varying system (l.c.t.v.s.) to be discretised is initially specified, it is of advantage to use a direct discretisation approach, i.e. to derive $\boldsymbol{H}(n, k)$ from $\boldsymbol{H}(t, \tau)$ and synthetise the discrete equivalent system from it, rather than synthetise first $\boldsymbol{H}(t, \tau)$ and then discretise the state equations. Here, the synthesis of a linear discrete-time-varying system (l.d.t.v.s.) from a specified $\boldsymbol{H}(n, k)$ is considered.

System synthesis from a specified $\boldsymbol{H}(n, k)$*:* The problem of synthetising an l.c.t.v.s. from its specified impulse-response matrix $\boldsymbol{H}(t, \tau)$ has been treated extensively in the literature.[1-5] The extension to the discrete-time case of some of the basic results, such as realisability conditions, equivalence transformations and reduced realisations,[1-3] is straightforward, and will be stated without proof.

Let S be an l.d.t.v.s. which is described by the state equations

$$\left.\begin{aligned} \boldsymbol{x}(n+1) &= \boldsymbol{A}(n)\,\boldsymbol{x}(n) + \boldsymbol{B}(n)\,\boldsymbol{u}(n) \\ \boldsymbol{y}(n) &= \boldsymbol{C}(n)\,\boldsymbol{x}(n) \end{aligned}\right\} \qquad (1)$$

where $\boldsymbol{x}(n)$ is an m-dimensional state vector, $\boldsymbol{u}(n)$ an r-dimensional input vector and $\boldsymbol{y}(n)$ a p-dimensional output vector. $\boldsymbol{A}(n)$, $\boldsymbol{B}(n)$ and $\boldsymbol{C}(n)$ are matrices of proper dimensions and n is the discrete time variable (an integer).

The triplet $\{\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n)\}$ is said to be a 'realisation' of S.

As in Reference 1, one obtains for the discrete-time case:

Theorem 1: A necessary and sufficient condition for the realisability of an impulse-response matrix $\boldsymbol{H}(n, k)$ by a system S of the form given by eqns. 1 is the existence of finite-dimensional matrices $\boldsymbol{F}(n)$ and $\boldsymbol{G}(k)$ such that

$$\boldsymbol{H}(n, k) = \boldsymbol{F}(n)\,\boldsymbol{G}(k) \qquad n_0 \leqslant k < n \qquad (2)$$

Corollary 1: Every $\boldsymbol{H}(n, k)$ in the form of eqn. 2 is realised by $\{\boldsymbol{I}, \boldsymbol{G}(n), \boldsymbol{F}(n)\}$. We call $\{\boldsymbol{I}, \boldsymbol{G}(n), \boldsymbol{F}(n)\}$ a *basic realisation.*

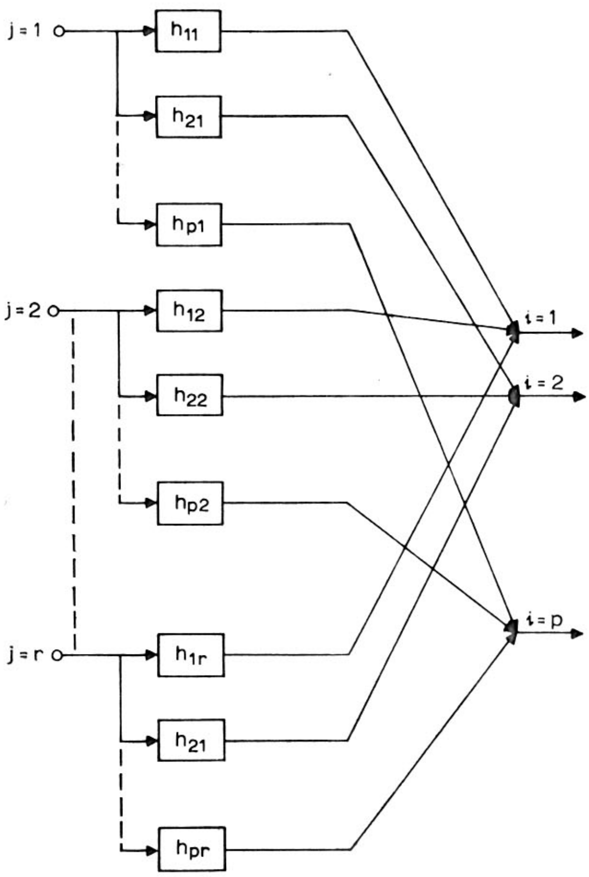

Fig. 1 *Crosscoupled realisation of a p × r impulse-response matrix* $\boldsymbol{H}(n, k) = [h_{ij}(n, k)]$

Remark: Equivalently, a basic realisation of $\boldsymbol{H}(t, \tau) = \boldsymbol{P}(t)\,\boldsymbol{Q}(\tau)$ is given by $\{\boldsymbol{O}, \boldsymbol{Q}(t), \boldsymbol{P}(t)\}$.

A linear transformation of the state vector $\boldsymbol{x}(n)$ of S to a new state vector $\boldsymbol{z}(n) = \boldsymbol{T}(n)\,\boldsymbol{x}(n)$, where $\boldsymbol{T}(n)$ is an $m \times m$ nonsingular matrix, results in an equivalent system S_T. The coefficient matrices of S_T are found, on substitution, to be

$$\boldsymbol{A}_T(n) = \boldsymbol{T}(n+1)\,\boldsymbol{A}(n)\,\boldsymbol{T}^{-1}(n) \qquad (3a)$$

$$\boldsymbol{B}_T(n) = \boldsymbol{T}(n+1)\,\boldsymbol{B}(n) \qquad \boldsymbol{C}_T(n) = \boldsymbol{C}(n)\,\boldsymbol{T}^{-1}(n) \qquad (3b)$$

and it is easily verified that $\boldsymbol{H}_T(n, k) = \boldsymbol{H}(n, k)$. Thus S and S_T are equivalent with respect to their impulse response (i.e. they are algebraically equivalent).[1]

By applying equivalence transformations to the basic realisation, infinitely many realisations can be generated. Furthermore, from Corollary 1 and eqn. 3*a*, we obtain

Theorem 2: A system of order m with an impulse-response matrix $\boldsymbol{H}(n, k) = \boldsymbol{F}(n)\,\boldsymbol{G}(k)$, where $\boldsymbol{F}(n)$ and $\boldsymbol{G}(k)$ are $p \times m$ and $m \times r$ matrices, respectively, can be realised in the form of eqn. 1 with any desired $m \times m$ nonsingular matrix $\boldsymbol{A}(n)$, by applying to the basic realisation $\{\boldsymbol{I}, \boldsymbol{G}(n), \boldsymbol{F}(n)\}$ an equivalence transformation $\boldsymbol{T}(n) = \boldsymbol{X}(n)$, where $\boldsymbol{X}(n)$ is a fundamental matrix corresponding to $\boldsymbol{A}(n)$ [i.e. $\boldsymbol{X}(n+1) = \boldsymbol{A}(n)\,\boldsymbol{X}(n)$].

Definition 1: A decomposition $\boldsymbol{F}(n)\,\boldsymbol{G}(k)$ of $\boldsymbol{H}(n, k)$ is said to be of order m if the number of columns in $\boldsymbol{F}(n)$ [= number of rows in $\boldsymbol{G}(k)$] is equal to m; or, equivalently, if the corresponding basic realisation is of order m.

Definition 2: A decomposition $\boldsymbol{F}(n)\,\boldsymbol{G}(k)$, its corresponding basic realisation and the equivalent realisations generated from it are said to be *globally* reduced[2, 3] if the columns of $\boldsymbol{F}(n)$ are linearly independent for $n \varepsilon (n_0, \infty)$, and so are the rows of $\boldsymbol{G}(n)$.

Theorem 3: Every realisable $\boldsymbol{H}(n, k)$ possesses a globally reduced realisation.

The proof to this theorem is as given in References 2 and 3 for the continuous-time case. Furthermore, the constructive nature of the proof yields a general method for reducing any given decomposition to a globally reduced one.

The course to be followed for synthetising a specified $\boldsymbol{H}(n, k) = \boldsymbol{F}(n)\,\boldsymbol{G}(k)$ is now clear. First, reduce the decomposition to a globally reduced one, then apply an equivalence transformation $\boldsymbol{T}(n)$ to the basic realisation to obtain, according to eqn. 3, other realisations.

It is of interest to note that, for the scalar case, the use of the *Casorati matrix*[6] as an equivalence transformation transforms the basic realisation to a canonical form [$\boldsymbol{A}_T(n)$ being the companion matrix].[7]

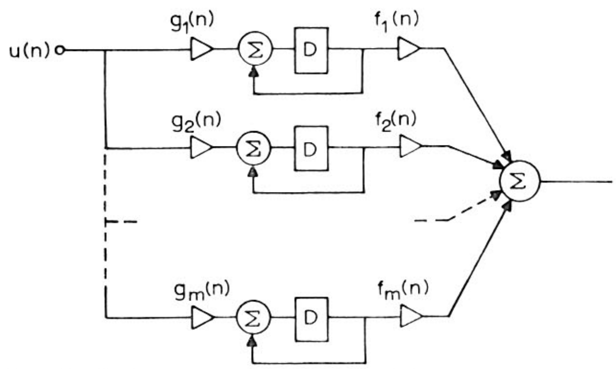

Fig. 2 *Basic realisation of* $h(n, k) = \boldsymbol{f}^T(n)\boldsymbol{g}(k)$

Decomposition of the impulse-response matrix: In the above synthesis procedure, it is assumed that $\boldsymbol{H}(n, k)$ is in a decomposed form. A problem arises, however, if $\boldsymbol{H}(n, k)$ is not given in such a form. It is possible to extend to the discrete case known decomposition methods which were obtained for l.c.t.v.s.,[4, 8] but, since they are rather complicated, a simple decomposition method is proposed below. Actually, the proposed method is also readily applicable to the continuous-time case and does not require derivatives of the elements of $\boldsymbol{H}(t, \tau)$, which are required by the known methods.

The proposed decomposition method is based on initially realising the $p \times r$ impulse-response matrix $\boldsymbol{H}(n, k) = [h_{ij}(n, k)]$ by the cross-coupled realisation shown in Fig. 1, and realising each scalar subsystem, having an impulse-response function* $h_{ij}(nk) = [\boldsymbol{f}^{ij}(n)]^T\,\boldsymbol{g}^{ij}(k)$, by the basic realisation $\{\boldsymbol{I}, \boldsymbol{g}^{ij}(n), [\boldsymbol{f}^{ij}(n)]^T\}$ of order m_{ij} (minimal) shown in Fig. 2. The above corresponds to the realisation of

* In the following, the superscript T denotes transpose

$H(n, k)$ by the basic realisation $\{I, G_0(n), F_0(n)\}$ of order

$$m_0 = \sum_{i=1}^{p} \sum_{j=1}^{r} m_{ij}$$

where

$$F_0(n) = \left[\begin{array}{ccc|ccc|c|ccc} f^{11\mathrm{T}} & & 0 & f^{12\mathrm{T}} & & 0 & & f^{1r\mathrm{T}} & & 0 \\ & f^{21\mathrm{T}} & & & f^{22\mathrm{T}} & & \cdots & & f^{2r\mathrm{T}} & \\ 0 & & \ddots & 0 & & \ddots & & & & \ddots \\ & & f^{p1\mathrm{T}} & & & f^{p2\mathrm{T}} & & & & f^{pr\mathrm{T}} \end{array}\right] \quad . \quad . \quad . \quad . \quad (4)$$

$$G_0(k) = \begin{bmatrix} g^{11\mathrm{T}} & g^{21\mathrm{T}} \ldots g^{p1\mathrm{T}} & 0 & 0 \ldots\ldots\ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots 0 & g^{12\mathrm{T}} \ldots g^{p2\mathrm{T}} & 0 \quad\quad 0 \ldots\ldots & 0 \\ 0 & 0 \ldots\ldots\ldots\ldots 0 & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 \ldots\ldots\ldots\ldots\ldots\ldots & 0 & g^{1r\mathrm{T}} \quad g^{2r\mathrm{T}} \ldots & g^{pr\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \quad . \quad . \quad . \quad . \quad (5)$$

and $F_0(n)$ and $G_0(k)$ are $p \times m_0$ and $m_0 \times r$ matrices, respectively. $F_0(n) G_0(k)$ is therefore an initial decomposition of order m_0 and can be written down from $H(n, k)$ by inspection. Furthermore, owing to the special form of the matrices $F_0(n)$ and $G_0(k)$, which permits the identification of linearly dependent columns in $F_0(n)$ and linearly dependent rows in $G_0(k)$ with almost no effort, a partial reduction in the order of the decomposition is easily obtained. This is done by discarding dependent rows in $G_0(k)$ [or columns in $F_0(n)$] so that the product of the resulting matrices remains $H(n, k)$. The final reduction (if necessary) to a globally reduced decomposition can be obtained by applying the procedure given in References 2 and 3.

Decomposition example: The following impulse-response matrix $H(n, k)$ is prescribed:

$$H(n, k) = \frac{1}{n^3 k^3} \begin{bmatrix} nk + n^2 k^2 & 2nk + n^2/k \\ 2nk + n^2 k^2 + nk^2 & 4nk + n^2/k + n/k \end{bmatrix} \quad (6)$$

An initial decomposition is found by inspection (according to eqns. 4 and 5) to be

$$F_0(n) = \frac{1}{n^3} \begin{bmatrix} n & n^2 & 0 & 0 & 0 & 2n & n^2 & 0 & 0 & 0 \\ 0 & 0 & 2n & n^2 & n & 0 & 0 & 4n & n^2 & n \end{bmatrix} \quad . \quad . \quad . \quad (7)$$

$$G_0(k) = \begin{bmatrix} k & k^2 & k & k^2 & k^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k & 1/k & k & 1/k & 1/k \end{bmatrix}^{\mathrm{T}} \quad . \quad . \quad . \quad (8)$$

Starting with $G_0(k)$, discarding in it proper rows and combining accordingly columns in $F_0(n)$, we obtain the decomposition

$$H(n, k) = F_1(n) G_1(k) = \frac{1}{n^3 k^3} \begin{bmatrix} n & n^2 & 2n & n^2 \\ n^2 & n^2 + n & 4n & n^2 + n \end{bmatrix}$$

$$\times \begin{bmatrix} k & k^2 & 0 & 0 \\ 0 & 0 & k & 1/k \end{bmatrix}^{\mathrm{T}} \quad (9)$$

Since dependent columns in $F_1(n)$ are also now easily identified, we proceed as above to obtain

$$H(n, k) = F_2(n) G_2(k)$$

$$= \frac{1}{n^3 k^3} \begin{bmatrix} n & n \\ 2n & n^2 + n \end{bmatrix} \begin{bmatrix} k & 2k \\ k^2 & 1/k \end{bmatrix} \quad . \quad . \quad (10)$$

which is only of order 2 and, in this case, also globally reduced, so that no further operations are necessary.

Conclusions: The fundamental results necessary for the synthesis of a linear discrete-time-varying system from a specified impulse-response matrix have been extended from known results for continuous-time systems. A simple decomposition method of the impulse-response matrix which is applicable to both discrete and continuous-time systems is proposed. A direct discretisation approach can be implemented by deriving a specification for $H(n, k)$, e.g. by using the approach of response equivalence[9] with respect to a certain desired input, and then applying the above results.

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