SYNTHESIS OF LINEAR DISCRETE-
TIME-VARYING SYSTEMS

Indexing terms: Discrete-time systems, Linear systems. Step
response, Control-system synthesis

The synthesis of a linear discrete-time-varying system from its
specified impulse-response matrix $H(n, k)$ is considered. The
results include a direct extension of those obtained for the
continuous-time counterpart problem, and a simple decom-
position method of $H(n, k)$ which is readily extendable to the
continuous-time case.

Introduction: A notable advantage of replacing a continu-
ous-time-varying system by a discrete equivalent one is the sim-
plexity with which the latter can be realised, because the time-
varying properties are obtained just by varying the multi-
plying coefficients with time. If the impulse-response matrix
$H(t, \tau)$ of the linear continuous-time-varying system (l.c.t.v.s.)
to be discretised is initially specified, it is of advantage to use
a direct discretisation approach, i.e. to derive $H(n, k)$ from
$H(t, \tau)$ and synthetise the discrete equivalent system from it,
rather than synthetise first $H(t, \tau)$ and then discretise the state
equations. Here, the synthesis of a linear discrete-time-
varying system (l.d.t.v.s.) from a specified $H(n, k)$ is con-
sidered.

System synthesis from a specified $H(n, k)$: The problem of
synthetising an l.c.t.v.s. from its specified impulse-response
matrix $H(t, \tau)$ has been treated extensively in the literature.$^1-^5$
The extension to the discrete-time case of some of the basic
results, such as realisability conditions, equivalence trans-
formations and reduced realisations,$^1-^3$ is straightforward,
and will be stated without proof.
Let $S$ be an I.d.t.v.s. which is described by the state equations

\[
\begin{align*}
x(n+1) &= A(n) x(n) + B(n) u(n) \\
y(n) &= C(n) x(n)
\end{align*}
\]

where $x(n)$ is an $m$-dimensional state vector, $u(n)$ an $r$-dimensional input vector and $y(n)$ a $p$-dimensional output vector. $A(n)$, $B(n)$ and $C(n)$ are matrices of proper dimensions and $n$ is the discrete time variable (an integer).

The triplet $(A(n), B(n), C(n))$ is said to be a 'realisation' of $S$.

As in Reference 1, one obtains for the discrete-time case:

**Theorem 1:** A necessary and sufficient condition for the realisability of an impulse-response matrix $H(n, k)$ by a system $S$ of the form given by eqns. 1 is the existence of finite-dimensional matrices $F(n)$ and $G(k)$ such that

\[
H(n, k) = F(n) G(k) \quad n_0 \leq k < n
\]

**Corollary 1:** Every $H(n, k)$ in the form of eqn. 2 is realised by $(I, G(n), F(n))$. We call $(I, G(n), F(n))$ a basic realisation.

**Definition 1:** A decomposition $F(n) G(k)$ of $H(n, k)$ is said to be of order $m$ if the number of columns in $F(n)$ $= \text{number of rows in } G(k)$ is equal to $m$; or, equivalently, if the corresponding basic realisation is of order $m$.

**Definition 2:** A decomposition $F(n) G(k)$, its corresponding basic realisation and the equivalent realisations generated from it are said to be globally reduced$^{2,3}$ if the columns of $F(n)$ are linearly independent for $m(n_0, \tau)$, and so are the rows of $G(n)$.

**Theorem 3:** Every realisable $H(n, k)$ possesses a globally reduced realisation.

The proof to this theorem is as given in References 2 and 3 for the continuous-time case. Furthermore, the constructive nature of the proof yields a general method for reducing any given decomposition to a globally reduced one.

The course to be followed for synthesising a specified $H(n, k) = F(n) G(k)$ is now clear. First, reduce the decomposition to a globally reduced one, then apply an equivalence transformation $T(n)$ to the basic realisation to obtain, according to eqn. 3, other realisations.

It is of interest to note that, for the scalar case, the use of the *Casorati matrix* as an equivalence transformation forms the basic realisation to a canonical form $[A_T(n)$ being the companion matrix].

**Decomposition of the impulse-response matrix:** In the above synthesis procedure, it is assumed that $H(n, k)$ is in a decomposed form. A problem arises, however, if $H(n, k)$ is not given in such a form. It is possible to extend to the discrete case known decomposition methods which were obtained for I.c.t.v.s.$^4,5,6$ but, since they are rather complicated, a simple decomposition method is proposed below. Actually, the proposed method is also readily applicable to the continuous-time case and does not require derivatives of the elements of $H(t, \tau)$, which are required by the known methods.

The proposed decomposition method is based on initially realising the $p \times r$ impulse-response matrix $H(n, k) = [h_{i,j}(n, k)]$ by the cross-coupled realisation shown in Fig. 1, and realising each scalar subsystem, having an impulse-response function $h_{i,j}(n, k) = [f_i(t)]^{\top} g_j(k)$, by the basic realisation $[I, f_i(t), [f_i(t)]^{\top}, [f_i(t)]^{\top}]$ of order $m_i$, (minimal) shown in Fig. 2. The above corresponds to the realisation of

\[
\begin{align*}
A_T(n) &= T(n+1) A(n) T^{-1}(n) \\
B_T(n) &= T(n+1) B(n) \\
C_T(n) &= C(n) T^{-1}(n)
\end{align*}
\]

and it is easily verified that $H_T(n, k) = H(n, k)$. Thus S and $S_T$ are equivalent with respect to their impulse response (i.e. they are algebraically equivalent).
$H(n, k)$ by the basic realisation: $L G_1(n), F_\ell(n)$, of order $m_\ell = \sum_{i=1}^\ell \sum_{i=1}^{m_i}$, where

$$F_\ell(n) = \begin{bmatrix} f_{11}^\ell & 0 & \cdots & 0 \\ f_{21}^\ell & f_{11}^\ell & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ f_{p1}^\ell & \cdots & 0 & f_{11}^\ell \\ \end{bmatrix} \cdots \begin{bmatrix} f_{11}^\ell \end{bmatrix}$$

(4)

$$G_\ell(n) = \begin{bmatrix} g_{11}^\ell & g_{12}^\ell & \cdots & g_{1r}^\ell \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{11}^\ell \end{bmatrix} \cdots \begin{bmatrix} g_{1r}^\ell \\ \end{bmatrix}$$

(5)

and $F_\ell(n)$ and $G_\ell(n)$ are $p \times m_\ell$ and $m_\ell \times r$ matrices, respectively. $F_\ell(n) G_\ell(n)$ is therefore an initial decomposition of order $m_\ell$ and can be written down from $H(n, k)$ by inspection. Furthermore, owing to the special form of the matrices $F_\ell(n)$ and $G_\ell(n)$, which permits the identification of linearly dependent columns in $F_\ell(n)$ and linearly dependent rows in $G_\ell(n)$ with almost no effort, a partial reduction in the order of the decomposition is easily obtained. This is done by discarding dependent rows in $G_\ell(n)$ [or columns in $F_\ell(n)$] so that the product of the resulting matrices remains $H(n, k)$. The final reduction (if necessary) to a globally reduced decomposition can be obtained by applying the procedure given in References 2 and 3.

**Decomposition example:** The following impulse-response matrix $H(n, k)$ is prescribed:

$$H(n, k) = \begin{bmatrix} n^2 + n^3 k & 2n^2 + nk^2 \\ 2nk + n^3 k^2 & 2nk^2 + n^2 k + nk \\ \end{bmatrix}$$

(6)

An initial decomposition is found by inspection (according to eqns. 4 and 5) to be

$$F_\ell(n) = \begin{bmatrix} n & n^2 & 0 & 0 & 0 \\ 0 & 2n & 0 & n^2 & 0 \\ 0 & 0 & n^2 & 2n & 0 \\ \end{bmatrix}$$

(7)

$$G_\ell(k) = \begin{bmatrix} k & k^{1/2} & k^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 1/k & k^{1/2} \\ \end{bmatrix}$$

(8)

Starting with $G_\ell(k)$, discarding in it proper rows and combining accordingly columns in $F_\ell(n)$, we obtain the decomposition

$$H(n, k) = F_\ell(n) G_\ell(k) = \frac{1}{n^3 k^3} \begin{bmatrix} n & 2n \\ n^2 & n^2 + n \\ 2n & 4n \\ \end{bmatrix} \begin{bmatrix} k & k^{1/2} & 0 & 0 \\ 0 & 0 & k & 1/k \\ \end{bmatrix}$$

(9)

Since dependent columns in $F_\ell(n)$ are also now easily identified, we proceed as above to obtain

$$H(n, k) = F_\ell(n) G_\ell(k) = \frac{1}{n^3 k^3} \begin{bmatrix} n & 2n \\ n^2 & n^2 + n \\ 2n & 4n \\ \end{bmatrix} \begin{bmatrix} k & k^{1/2} & 0 & 0 \\ 0 & 0 & k & 1/k \\ \end{bmatrix}$$

(10)

which is only of order 2 and, in this case, also globally reduced, so that no further operations are necessary.

**Conclusions:** The fundamental results necessary for the synthesis of a linear discrete-time-variating system from a specified impulse-response matrix have been extended from known results for continuous-time systems. A simple decom-