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Reduction and transformation of linear discrete-time-varying systems†

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The purpose of this paper is to extend to the discrete-time case techniques for order reduction and equivalence transformation to canonical forms of time-varying systems.

The extension is based on particular controllability and observability matrices which are completely analogous, with respect to their transformational properties, to their continuous-time counterparts.

1. Introduction

The digital simulation of a continuous-time system requires its discretization, which in general amounts to the derivation of a set of difference equations from the differential equations describing the system. Discretization methods, however, do not necessarily preserve canonical forms or even the order of the original system [e.g. if high-order difference schemes are used (Kelly 1967)]. The discretization procedure is satisfactorily completed only if the discretized system is of the lowest possible order (reduced) and preferably in a canonical form. For linear continuous-time-varying systems (l.c.t.v.s.) the problem of system reduction has been solved (Stubberud 1963, Silverman and Meadows 1965 a, b, Glass and D'Angelo 1967, Albertson and Womack 1968) by utilizing transformational properties of particular controllability and observability matrices which are expressed in a recursive form in terms of the coefficient matrices of the system (Stubberud 1964, Silverman and Meadows 1965 b, 1967, Chang 1965). The same matrices [and a modified one (Chao and Liu 1971)] have also been utilized in deriving equivalence transformations to particular canonical forms (Silverman and Meadows 1965 a, Silverman 1966, Ramaswami and Ramar 1969). Some of the above results are extended in the sequel to the discrete-time case by utilizing proper controllability and observability matrices which are shown to have analogous transformational properties to their continuous-time counterparts.

2. Algebraic criteria for controllability and observability

The linear discrete-time-varying system (l.d.t.v.s.) S under consideration is described by

$$S: \quad \mathbf{x}(n+1) = \mathbf{A}(n)\mathbf{x}(n) + \mathbf{B}(n)\mathbf{u}(n), \tag{1}$$

$$\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n), \tag{2}$$

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where the state $\mathbf{x}(n)$ is an m-vector, the input $\mathbf{u}(n)$ an r-vector, and the output $\mathbf{y}(n)$ a p-vector. $\mathbf{A}(n)$, $\mathbf{B}(n)$ and $\mathbf{C}(n)$ are matrices of proper dimensions, and n is the discrete time variable (an integer).

2.1. *Controllability*

The system S is said to be completely controllable if any initial state can be transferred to any other state in a finite time. It is said to be totally controllable if the desired state can be attained from any initial state in q time instants, where q is any fixed integer. S is then also said to be totally q-controllable.

Repeated application of (1) yields

$$\begin{aligned}\mathbf{x}(n+q)-\mathbf{A}(n+q-1)\mathbf{A}(n+q-2)\ldots\mathbf{A}(n)\mathbf{x}(n)\\ =\mathbf{A}(n+q-1)\mathbf{A}(n+q-2)\ldots\mathbf{A}(n+1)\mathbf{B}(n)\mathbf{u}(n)\\ +\mathbf{A}(n+q-1)\ldots\mathbf{A}(n+2)\mathbf{B}(n+1)\mathbf{u}(n+1)\\ +\ldots+\mathbf{B}(n+q-1)\mathbf{u}(n+q-1),\end{aligned} \tag{3}$$

and hence the system S is completely controllable if and only if there exists an integer q, for every n, such that the matrix $Q_c(n, q)$ defined below (controllability matrix) has rank m :

$$\begin{aligned}\mathbf{Q}_c(n,q)\triangleq[\mathbf{B}(n),\ \mathbf{A}^{-1}(n+1)\mathbf{B}(n+1),\\ \ldots,\ \mathbf{A}^{-1}(n+1)\mathbf{A}^{-1}(n+2)\ldots\mathbf{A}^{-1}(n+q-1)\mathbf{B}(n+q-1)].\end{aligned} \tag{4}$$

Alternatively, one obtains from eqn. (3) the modified controllability matrix

$$\begin{aligned}\mathbf{Q}_c^*(n,q)\triangleq[\mathbf{A}(n+q-1)\ldots\mathbf{A}(n+1)\mathbf{B}(n),\\ \ldots,\ \mathbf{A}(n+q-1)\mathbf{B}(n+q-2),\ \mathbf{B}(n+q-1)].\end{aligned} \tag{5}$$

A system S which is completely controllable is also totally controllable (totally q-controllable) if the above rank condition for the controllability matrices is satisfied with some fixed integer q.

Note that for time-invariant systems $\mathbf{Q}_c^*(n, q)$ yields the well-known result $[\mathbf{A}^{q-1}\mathbf{B}, \ldots, \mathbf{AB}, \mathbf{B}]$.

It is convenient to express $\mathbf{Q}_c(n, q)$ and $\mathbf{Q}_c^*(n, q)$ in the following recursive forms :

$$\left.\begin{aligned}&\mathbf{Q}_c(n,q)\triangleq[\mathbf{P}_0(n),\ \mathbf{P}_1(n),\ \ldots,\ \mathbf{P}_{q-1}(n)],\\ \text{where}\quad &\mathbf{P}_0(n)=\mathbf{B}(n)\ ;\quad \mathbf{P}_{i+1}(n)=\mathbf{A}^{-1}(n+1)\mathbf{P}_i(n+1),\end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned}&\mathbf{Q}_c^*(n,q)\triangleq[\mathbf{P}_0^*(n),\ \mathbf{P}_1^*(n),\ \ldots,\ \mathbf{P}_{q-1}^*(n)],\\ \text{where}\quad &\mathbf{P}_{q-1}^*(n)=\mathbf{B}(n+q-1)\ ;\quad \mathbf{P}_i^*(n)=\mathbf{A}(n+q-1)\mathbf{P}_{i+1}^*(n-1).\end{aligned}\right\} \tag{7}$$

2.2. *Observability*

The system S is said to be completely observable if the state $\mathbf{x}(n)$ can be determined, for every n, from knowledge of the input $\mathbf{u}(n)$ and the output $\mathbf{y}(n)$ over a finite time sequence $[n, n+q]$. It is said to be totally observable if the integer q above is fixed for all n. S is then also said to be totally q-observable.

By applying the dual system concept (Sarachik and Kreindler 1965) the observability matrices below are obtained.

The dual to $\mathbf{Q}_c(n, q)$ of (6) is

$$\left.\begin{aligned} &\mathbf{Q}_0(n, q) \triangleq [\mathbf{S}_0(n), \mathbf{S}_1(n), \ldots, \mathbf{S}_{q-1}(n)], \\ \text{where} \quad &\mathbf{S}_0(n) = \mathbf{C}^{\mathrm{T}}(n+1)\ ; \quad \mathbf{S}_{i+1}(n) = \mathbf{A}^{\mathrm{T}}(n+1)\mathbf{S}_i(n+1) \end{aligned}\right\} \tag{8}$$

and where the superscript T denotes ' transpose '.

Similarly, the dual to $\mathbf{Q}_c{}^*(n, q)$ of (7) is

$$\left.\begin{aligned} &\mathbf{Q}_c{}^*(n, q) = [\mathbf{S}_0{}^*(n), \mathbf{S}_1{}^*(n), \ldots, \mathbf{S}_{q-1}{}^*(n)], \\ \text{where} \quad &\mathbf{S}_{q-1}{}^*(n) = \mathbf{C}^{\mathrm{T}}(n+q)\ ; \quad \mathbf{S}_i{}^*(n) = [\mathbf{A}^{\mathrm{T}}(n+q-1)]^{-1}\mathbf{S}_{i+1}{}^*(n-1). \end{aligned}\right\} \tag{9}$$

For the special case of time-invariant systems $\mathbf{Q}_0(n, q)$ yields the well-known result $[\mathbf{C}^{\mathrm{T}}, \mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}, \ldots, (\mathbf{A}^{\mathrm{T}})^{q-1}\mathbf{C}^{\mathrm{T}}]$.

3. Equivalent systems

Definition 1. A system S_{T} is said to be equivalent to the system S of (1) if its state vector $\mathbf{x}_{\mathrm{T}}(n)$ is obtained from $\mathbf{x}(n)$ by $\mathbf{x}_{\mathrm{T}}(n) = \mathbf{T}(n)\mathbf{x}(n)$, where $\mathbf{T}(n)$ is an $m \times m$ non-singular matrix, said to be the *equivalence transformation* which transforms S to S_{T}.

Let $\{\mathbf{A}_{\mathrm{T}}(n), \mathbf{B}_{\mathrm{T}}(n), \mathbf{C}_{\mathrm{T}}(n)\}$ be the coefficient matrices of S_{T}. Then, substitution of $\mathbf{x}_{\mathrm{T}}(n) = \mathbf{T}(n)\mathbf{x}(n)$ in eqn. (1) yields

$$\mathbf{A}_{\mathrm{T}}(n) = \mathbf{T}(n+1)\mathbf{A}(n)\mathbf{T}^{-1}(n)\ ; \quad \mathbf{B}_{\mathrm{T}}(n) = \mathbf{T}(n+1)\mathbf{B}(n)\ ; \quad \mathbf{C}_{\mathrm{T}}(n) = \mathbf{C}(n)\mathbf{T}^{-1}(n). \tag{10}$$

The relation between fundamental matrices of equivalent systems is then found to be $\mathbf{\Phi}_{\mathrm{T}}(n, k) = \mathbf{T}(n)\mathbf{\Phi}(n, k)\mathbf{T}^{-1}(n)$, from which it is easily verified that the impulse response matrices of equivalent systems are identical. The above equivalence is therefore with respect to the input–output properties (algebraic equivalence).

The transformational properties of the controllability and observability matrices presented in the previous section can now be examined.

Let $\mathbf{Q}_{c\mathrm{T}}(n, q)$ and $\mathbf{Q}_{0\mathrm{T}}(n, q)$ denote the controllability and observability matrices, respectively, of S_{T}. Then, by substitution of eqn. (10) into eqns. (6) and (8) and applying induction, one obtains the relations

$$\left.\begin{aligned} \mathbf{Q}_{c\mathrm{T}}(n, q) &= \mathbf{T}(n+1)\mathbf{Q}_c(n, q), \\ \mathbf{Q}_{0\mathrm{T}}(n, q) &= [\mathbf{T}^{\mathrm{T}}(n+1)]^{-1}\mathbf{Q}_0(n, q). \end{aligned}\right\} \tag{11}$$

Similarly, substitution of eqn. (10) into eqns. (7) and (9) yields

$$\left.\begin{aligned} \mathbf{Q}_{c\mathrm{T}}{}^*(n, q) &= \mathbf{T}(n+q)\mathbf{Q}_c{}^*(n, q), \\ \mathbf{Q}_{0\mathrm{T}}{}^*(n, q) &= [\mathbf{T}^{\mathrm{T}}(n+q)]^{-1}\mathbf{Q}_0{}^*(n, q). \end{aligned}\right\} \tag{12}$$

It is also noted, from eqn. (11), that

$$\mathbf{Q}_{0\mathrm{T}}{}^{\mathrm{T}}(n, q)\mathbf{Q}_{c\mathrm{T}}(n, q) = \mathbf{Q}_0{}^{\mathrm{T}}(n, q)\mathbf{Q}_c(n, q). \tag{13}$$

A similar result to (13) holds for the modified matrices $\mathbf{Q}_c^*(n, q)$ and $\mathbf{Q}_0^*(n, q)$. From eqn. (13) the matrix $\mathbf{\Psi}(n, q) \triangleq \mathbf{Q}_0^{\mathrm{T}}(n, q)\mathbf{Q}_c(n, q)$ is clearly invariant under equivalence transformations. $\mathbf{\Psi}(n, q)$ can therefore be used for checking if given systems are equivalent.

The above transformational properties are completely analogous to those obtained for the continuous-time counterparts (Silverman and Meadows 1965 a, 1966). The importance of this analogy is in the ability of readily extending, to the discrete-time case, results on system reduction and equivalence transformations to canonical forms. Other known matrices (Grammaticos 1969) do not have such a complete analogy in transformational properties, and cannot be applied, readily, to the above problems.

4. Application to system reduction

The extension to the discrete-time case of reduction techniques will be limited here to the problem considered by Silverman and Meadows (1965 a)—systems reduction ' from the input '. Yet, other results (e.g. those of Silverman and Meadows 1966, and D'Angelo 1970) can be extended in the same way.

Definition 2. System (1) is *reducible from the input to* order $\mu_c \leqslant m$ and to no lower order if there exists an equivalence transformation $\mathbf{z}(n) = \mathbf{T}_c(n)\mathbf{x}(n)$ such that

$$\left.\begin{aligned} \hat{S}: \quad \mathbf{z}(n+1) &= \begin{bmatrix} \mathbf{z}_1(n+1) \\ \mathbf{z}_2(n+1) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_{11}(n) & \hat{\mathbf{A}}_{12}(n) \\ \mathbf{0} & \hat{\mathbf{A}}_{22}(n) \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(n) \\ \mathbf{z}_2(n) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{B}}_1(n) \\ \mathbf{0} \end{bmatrix} \mathbf{u}(n) \\ \mathbf{y}(n) &= \mathbf{y}_1(n) + \mathbf{y}_2(n) = [\hat{\mathbf{C}}_1(n)\hat{\mathbf{C}}_2(n)]\mathbf{z}(n) \end{aligned}\right\} \quad (14)$$

and the μ_cth-order subsystem $\hat{S}_1$ below is totally controllable :

$$\left.\begin{aligned} \hat{S}_1: \quad \mathbf{z}_1(n+1) &= \hat{\mathbf{A}}_{11}(n)\mathbf{z}_1(n) + \hat{\mathbf{B}}_1(n)\mathbf{u}(n), \\ \mathbf{y}_1(n) &= \hat{\mathbf{C}}_1(n)\mathbf{z}_1(n). \end{aligned}\right\} \quad (15)$$

The reduction procedure constitutes here the determination of $\mathbf{T}_c(n)$ which transforms S of (1) to $\hat{S}$ of (14), from which $\hat{S}_1$—the reduced system—is readily identified.

The controllability matrix $\hat{\mathbf{Q}}_c(n, q)$ corresponding to $\hat{S}$ is found by using eqns. (6) and (14). Since eqn. (6) involves $\hat{\mathbf{A}}^{-1}(n)$ the matrix inversion identity given in Appendix 1 is applied. The result is that the last $m - \mu_c$ rows of $\hat{\mathbf{Q}}_c(n, q)$ are zero. The above and (11) yield

$$\hat{\mathbf{Q}}_c(n, q) = \mathbf{T}_c(n+1)\mathbf{Q}_c(n, q) = \begin{bmatrix} \hat{\mathbf{Q}}_{c1}(n, q) \\ \mathbf{0} \end{bmatrix} \quad (16)$$

in which $\hat{\mathbf{Q}}_{c1}(n, q)$ has μ_c rows.

A similar result is obtained (without requiring a matrix inversion) by using $\mathbf{Q}_c^*(n, q)$ of (7). The following theorem provides a condition under which an equivalence transformation $\mathbf{T}_c(n)$ which satisfies (16) reduces the system.

Theorem 1. If an equivalence transformation $\mathbf{T}_c(n)$ exists such that (16) is satisfied, and $\hat{\mathbf{Q}}_{c1}(n, q)$ has rank μ_c for all n and all $q \geqslant q_c$, where q_c is a fixed integer ($q_c \geqslant m/r$), then system S is reducible to a μ_cth-order totally controllable subsystem (totally q_c-controllable).

This theorem is an extension of a similar one obtained by Silverman and Meadows (1965 a) for l.c.t.v.s. and the proof, given in Appendix 2, is a modification of it to this case.

Finally, an explicit formulation for the reduction of a class of systems is given by the following theorem :

Theorem 2. If $\mathbf{Q}_c(n, q)$ has rank $\mu_c < m$ for all n and all $q \geqslant q_c$, where q_c is a fixed integer ($q_c \geqslant m/r$), and a submatrix of $\mathbf{Q}_c(n, q)$ also has rank μ_c, for the above conditions, then the system S is not controllable and can be reduced to a totally controllable (totally q_c-controllable) subsystem of order μ_c, by applying the equivalence transformation $\mathbf{T}_c(n)$ given by (17) below :

$$\left.\begin{aligned} \mathbf{T}_c(n+1) &= \begin{bmatrix} \mathbf{I}_{\mu_c} & \mathbf{0} \\ -\mathbf{Q}_{c2}(n, q_c)\mathbf{Q}_{c1}{}^{\#}(n, q_c) & \mathbf{I}_{m-\mu} \end{bmatrix}, \\ \text{where} \quad \mathbf{Q}_{c1}{}^{\#} &= \mathbf{Q}_{c1}{}^{\mathrm{T}}(\mathbf{Q}_{c1}\mathbf{Q}_{c1}{}^{\mathrm{T}})^{-1} \end{aligned}\right\} \tag{17}$$

is a generalized inverse (Penrose 1955, Greville 1959) of $\mathbf{Q}_{c1}$ and the rows of $\mathbf{Q}_c(n, q)$ have been reordered so that

$$\mathbf{Q}_c(n, q) = \begin{bmatrix} \mathbf{Q}_{c1}(n, q) \\ \mathbf{Q}_{c2}(n, q) \end{bmatrix}. \tag{18}$$

The proof of the above theorem is by construction and is completely identical to the one given for l.c.t.v.s. (Silverman and Meadows 1965 a) and hence is omitted. If one uses $\mathbf{Q}_c{}^*(n, q)$ instead of $\mathbf{Q}_c(n, q)$ a similar expression to (17) is obtained, but with $\mathbf{T}_c(n+q)$ on the left-hand side.

Example. A system L is described by

$$L: \quad \mathbf{x}(n+1) = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \mathbf{x}(n) + \begin{bmatrix} a_1{}^n \\ a_2{}^n \end{bmatrix} u(n), \tag{19}$$

where a_1 and a_2 are non-zero and unequal constants.

Applying eqn. (6), one obtains

$$\mathbf{Q}_c(n, q) = \begin{bmatrix} a_1{}^n & -a_1{}^n & & (-1)^{q-1}a_1{}^n \\ & & \cdots & \\ a_2{}^n & -a_2{}^n & & (-1)^{q-1}a_2{}^n \end{bmatrix}, \tag{20}$$

and since rank $\mathbf{Q}_c(n, q) = 1$, for all n and all $q \geqslant m/r = 2$, the system is not controllable.

To apply Theorem 2 we use $\mathbf{Q}_c(n, 2)$, for which : $\mu_c = 1$; $\mathbf{Q}_{c1} = [a_1{}^n, -a_1{}^n]$, $\mathbf{Q}_{c2} = [a_2{}^n, -a_2{}^n]$ and $\mathbf{Q}_{c1}{}^{\#} = [(a_1/2)^n, (-a_1/2)^n]^{\mathrm{T}}$. Hence, by eqn. (17),

$$\mathbf{T}_c(n+1) = \begin{bmatrix} 1 & 0 \\ -(a_2/a_1)^n & 1 \end{bmatrix}. \tag{21}$$

The ceefficient matrices of the equivalent system (of form (14)) are now found, by applying eqn. (21) to eqn. (10), to be

$$\hat{\mathbf{A}} = \mathbf{A} = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}; \quad \hat{\mathbf{B}}(n) = \begin{bmatrix} a_1{}^n \\ 0 \end{bmatrix}, \tag{22}$$

and hence the reduced system (first order) is described by

$$\hat{x}(n+1) = -a_1\hat{x}(n) + a_1{}^n u(n). \tag{23}$$

5. Application to system transformation

The relations obtained in eqns. (11) and (12) provide the means for an explicit determination of the equivalence transformation which relates two equivalent systems S and S_{T}. Thus, if two totally q_{c}-controllable systems S and S_{T} are equivalent, the equivalence transformation which relates them is obtained from eqn. (11) as

$$\mathbf{T}(n+1) = \mathbf{Q}_{\mathrm{cT}}(n, q_{\mathrm{c}})\mathbf{Q}_{\mathrm{c}}{}^{\#}(n, q_{\mathrm{c}}) \tag{24}$$

similarly, if two totally q_0-observable systems S and S_{T} are equivalent we obtain

$$\mathbf{T}(n+1) = \{[\mathbf{Q}_{0\mathrm{T}}(n, q_0)\mathbf{Q}_0{}^{\#}(n, q_0)]^{-1}\}^{\mathrm{T}}. \tag{25}$$

Use of eqn. (12) yields similar results in terms of the modified matrices.

For some particular canonical forms the controllability or observability matrice can be found explicitly even if the coefficient matrices are not known explicitly. In such cases the equivalence transformation of a given system to the particular canonical form can be determined by use of eqn. (24) or (25), as shown in the two cases below, which for simplicity consider scalar systems (single input-single output).

A widely used canonical form for scalar systems is

$$S_0: \quad \left.\begin{aligned} \mathbf{z}(n+1) &= \begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \vdots \\ \alpha_1(n) & \alpha_2(n) & \ldots\ldots \alpha_m(n) \end{bmatrix} \mathbf{z}(n) + \begin{bmatrix} b_1(n) \\ b_2(n) \\ b_m(n) \end{bmatrix} u(n), \\ y(n) &= [1 \quad 0 \ldots 0]\mathbf{z}(n). \end{aligned}\right\} \tag{26}$$

Substitution of the coefficient matrices of S_0 above into eqn. (8) yields the following form for $\mathbf{Q}_{0\mathrm{T}}(n, q)$:

$$\mathbf{Q}_{0\mathrm{T}}(n, q) = [\mathbf{I}_m \vdots \mathbf{Q}]; \quad q \geqslant m, \tag{27}$$

where $\mathbf{Q}$ has $q-m$ columns and $\mathbf{I}_m$ is the $m \times m$ identity matrix.

Clearly, rank $\mathbf{Q}_{0\mathrm{T}}(n, q) = m$, for all m and any $q \geqslant m$, which means that system S_0 is totally m-observable. Any other system S can, therefore, be equivalent to S_0 only if it is totally m-observable.

If a system S is known to be equivalent to S_0 then the relating equivalence transformation is found from eqns. (25) and (27) (with $q=m$) to be

$$\mathbf{T}(n+1)=\{[\mathbf{Q}_0{}''(n, m)]^{-1}\}^{\mathrm{T}}=\mathbf{Q}_0{}^{\mathrm{T}}(n, m), \tag{28}$$

which is explicitly given in terms of the coefficient matrices of S. Furthermore, as proved in Appendix 3,

Theorem 3. A necessary and sufficient condition that system S of (1), with $r=p=1$, be equivalent to S_0 of (20) is that S be totally m-observable. If it is so, the equivalence transformation is given by eqn. (28).

It is of interest to note that for the continuous-time case (Silverman and Meadows 1965 a) the necessary and sufficient condition in the counterpart to Theorem 3 is *uniform observability*. Thus, for discrete as well as for continuous-time-systems the above conditions are stronger than total observability.

Another useful canonical form is given by

$$\left.\begin{aligned} S_{\mathrm{c}}: \quad \mathbf{v}(n+1) &= \begin{bmatrix} \alpha_1(n) & 1 & 0 \ldots 0 \\ \alpha_2(n) & 0 & 1 \ldots 0 \\ \vdots & & \vdots \\ \vdots & & 1 \\ \alpha_m(n) & 0 & 0 \quad 0 \end{bmatrix} \mathbf{v}(n) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(n) \\ y(n) &= [c_1(n)c_2(n) \ldots c_m(n)]\mathbf{v}(n). \end{aligned}\right\} \tag{29}$$

In this case, if we consider the modified controllability matrix $\mathbf{Q}_{\mathrm{c}}{}^*(n, q)$ of eqn. (7), we find for system S_{c} of (29) that

$$\mathbf{Q}_{\mathrm{cT}}{}^*(n, q)=[\mathbf{I}_m \vdots \mathbf{Q}^*] ; \quad q \geqslant m. \tag{30}$$

Clearly, S_{c} is totally m-controllable. The following theorem can now be stated

Theorem 4. A necessary and sufficient condition that system S of (1), with $r=p=1$, be equivalent to system S_{c} of (29) is that S be totally m-controllable. If it is so, the equivalence transformation is given by

$$\mathbf{T}(n+m)=[\mathbf{Q}_{\mathrm{c}}{}^*(n, m)]^{-1}. \tag{31}$$

The proof to this theorem follows exactly the one given for Theorem 3 and therefore is omitted. Equation (31) is obtained from eqns. (12) and (30) (with $q=m$). In the continuous-time counterpart to Theorem 4 it is required that the system be *uniformly controllable* (Chao and Liu 1971). Thus, for both types of system, continuous and discrete the above necessary and sufficient conditions are stronger than total controllability.

6. Conclusions

The complete analogy in transformational properties between $\mathbf{Q}_{\mathrm{c}}(n, q)$ and $\mathbf{Q}_0(n, q)$ and their continuous-time counterparts enabled the extension to the discrete-time case of results on system reduction and equivalence transformation. The extension of other results, such as further treatment of reduction techniques (Glass and D'Angelo 1967, Silverman 1966 and Silverman and Meadows 1966), system stabilization (Wolovich 1968), construction of inverse systems (Silverman 1968, 1969) and other applications (Milo and Policastro 1970), may now be attempted in a similar way.

It has been shown constructively that a necessary and sufficient condition for the existence of an equivalence transformation which transforms a scalar system S, of order m, to the canonical form (26) or (29) is that S be totally m-observable or totally m-controllable, respectively. These results point to the analogy between uniform controllability and uniform observability for continuous-time systems and total m-controllability and total m-observability, respectively, for discrete-time systems.

Appendix 1

A matrix inversion identity

Consider the partitioned matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}, \quad \text{(A1 1)}$$

where $\mathbf{E}$ and $\mathbf{H}$ are assumed to be square non-singular matrices. The inverse of $\mathbf{V}$ is then given by (Fortmann 1970)

$$\mathbf{V}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{F}\mathbf{L}^{-1}\mathbf{G}\mathbf{E}^{-1} & -\mathbf{E}^{-1}\mathbf{F}\mathbf{L}^{-1} \\ -\mathbf{L}^{-1}\mathbf{G}\mathbf{E}^{-1} & \mathbf{L}^{-1} \end{bmatrix} \quad \text{(A1 2)}$$

in which $\mathbf{L}^{-1} = \mathbf{H} - \mathbf{G}\mathbf{E}^{-1}\mathbf{F}$.

Appendix 2

Proof of theorem 1

Let the coefficient matrices of the transformed system S be $[\hat{\mathbf{A}}(n), \hat{\mathbf{B}}(n), \hat{\mathbf{C}}(n)]$ and write

$$\hat{\mathbf{A}}(n) = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22} \end{bmatrix} : \quad \hat{\mathbf{B}}(n) = \begin{bmatrix} \hat{\mathbf{B}}_1 \\ \hat{\mathbf{B}}_2 \end{bmatrix}. \quad \text{(A2 1)}$$

Then, according to Definition 2 and eqn. (14), S will be reduced 'from the input' if $\hat{\mathbf{A}}_{21} = \mathbf{0}$ and $\hat{\mathbf{B}}_2 = \mathbf{0}$.

By hypothesis and by eqn. (6)

$$\hat{\mathbf{Q}}_c(n, q) = [\hat{\mathbf{Q}}_{c1}{}^{\mathrm{T}}(n, q), \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}} = [\hat{\mathbf{P}}_0(n), \ldots, \hat{\mathbf{P}}_{q-1}(n)],$$

and

$$\hat{\mathbf{P}}_0(n) = [\hat{\mathbf{B}}_1{}^{\mathrm{T}}, \hat{\mathbf{B}}_2{}^{\mathrm{T}}]^{\mathrm{T}}.$$

Hence $\hat{\mathbf{B}}_2(n) = \mathbf{0}$.

To show $\hat{\mathbf{A}}_{21}(n) = \mathbf{0}$, define $\mathbf{Q}_c'(n, q)$ as

$$\mathbf{Q}_c'(n, q) \triangleq [\mathbf{P}_1(n), \mathbf{P}_2(n), \ldots, \mathbf{P}_q(n)]. \quad \text{(A2 2)}$$

Comparison of eqns. (6) and (A2 2) shows

$$\mathbf{Q}_c'(n, q) = \mathbf{A}^{-1}(n+1)\,\mathbf{Q}_c(n+1, q) \quad \text{(A2 3)}$$

or

$$\mathbf{Q}_c(n+1, q) = \mathbf{A}(n+1)\mathbf{Q}_c'(n, q). \tag{A2 4}$$

Clearly, for $q \geq q_c$, rank $\mathbf{Q}_c'(n, q) = \text{rank } \mathbf{Q}_c(n, q) = \mu_c$.

Furthermore, since rank $\mathbf{Q}_c(n, q+1) = \text{rank } \mathbf{Q}_c(n, q)$, for $q \geq q_c$, and $\mathbf{Q}_c(n, q+1) = [\mathbf{Q}_c(n, q), \mathbf{P}_q(n)]$, the columns of $\mathbf{P}_q(n)$ are linear combinations of the columns of $\mathbf{Q}_c(n, q)$. Hence for $\hat{S}$:

$$\hat{\mathbf{P}}_q(n) = [\hat{\mathbf{P}}_{q1}{}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}},$$

and therefore

$$\hat{\mathbf{Q}}_c'(n, q) = [\mathbf{Q}_{c1}{}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}.$$

Using eqns. (A2 1) and (A2 4) we obtain

$$\begin{bmatrix} \hat{\mathbf{Q}}_{c1}(n+1, q) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_{11}(n+1) & \hat{\mathbf{A}}_{12}(n+1) \\ \hat{\mathbf{A}}_{21}(n+1) & \hat{\mathbf{A}}_{22}(n+1) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Q}}_{c1}' \\ \mathbf{0} \end{bmatrix} \tag{A2 5}$$

implying $\hat{\mathbf{A}}_{21}(n+1)\hat{\mathbf{Q}}_{c1}'(n, q) = \mathbf{0}$.

Since for $q \geq q_c$ rank $\hat{\mathbf{Q}}_{c1}(n, q) = \text{rank } \mathbf{Q}_{c1}(n, q) = \mu_c$,

for all n, we have $\hat{\mathbf{A}}_{21}(n) = \mathbf{0}$. Q.E.D.

Appendix 3

Proof of theorem 3

Necessity is clear from the discussion preceding the theorem.

To prove sufficiency one has to show that if system S of (1) (with $r = p = 1$) is totally m-observable there exists a $\mathbf{T}(n)$ which transforms S to S_0 of (26).

By hypothesis $\mathbf{Q}_0(n, m)$ is non-singular for all n and we may let

$$\mathbf{T}(n+1) = \mathbf{Q}_0{}^{\mathrm{T}}(n, m).$$

Applying eqn. (11)

$$\mathbf{Q}_{0\mathrm{T}}(n, m) = [\mathbf{T}^{\mathrm{T}}(n+1)]^{-1}\mathbf{Q}_0(n, m) = \mathbf{I}_m, \tag{A3 1}$$

and we have now to show that $\mathbf{Q}_{0\mathrm{T}}(n, m)$ of (A3 1) corresponds uniquely to $\mathbf{A}_{\mathrm{T}}(n)$ of eqn. (26).

Defining $\mathbf{Q}_0'(n, q) \triangleq [\mathbf{S}_1(n), \mathbf{S}_2(n), \ldots, \mathbf{S}_m(n)]$ and comparing with eqn. (8) we obtain

$$\mathbf{Q}_0'(n, q) = \mathbf{A}^{\mathrm{T}}(n+1)\mathbf{Q}_0(n+1, q). \tag{A3 2}$$

Applying eqns. (A3 1) and (A3 2) for determining $\mathbf{A}_{\mathrm{T}}(n)$ it is found to be uniquely determined by

$$\mathbf{A}_{\mathrm{T}}{}^{\mathrm{T}}(n+1) = \mathbf{Q}_{0\mathrm{T}}'(n, m) = [\mathbf{S}_{1\mathrm{T}}(n), \ldots, \mathbf{S}_{m\mathrm{T}}(n)]. \tag{A3 3}$$

Since $[\mathbf{S}_{0\mathrm{T}}(n), \ldots, \mathbf{S}_{m-1}(n)] = \mathbf{I}_m$, and by eqn. (11) $\mathbf{S}_{m\mathrm{T}}(n) = [\mathbf{T}^{\mathrm{T}}(n+1)]^{-1}\mathbf{S}_m(n)$, we obtain from eqn. (A3 3)

$$\mathbf{A}_{\mathrm{T}}{}^{\mathrm{T}}(n+1) = \begin{bmatrix} 0 & 0 \ldots 0 & \alpha_1(n+1) \\ 1 & 0 \ldots 0 & \alpha_2(n+1) \\ \vdots & \vdots \quad\ \vdots & \vdots \\ 0 & 0 \ldots 1 & \alpha_m(n+1) \end{bmatrix}, \tag{A3 4}$$

where

$$[\alpha_1(n+1), \alpha_2(n+1), \ldots, \alpha_m(n+1)] = \mathbf{S}_{mT}(n)$$
$$= [\mathbf{T}^T(n+1)]^{-1}\mathbf{S}_m(n) = \mathbf{Q}_0^{-1}(n, m)\mathbf{S}_m(n).$$

Q.E.D.

References

ALBERTSON, H. D., and WOMACK, W. D., 1968, *I.E.E.E. Trans. autom. Control*, **13**, 307.

CHANG, A., 1965, *I.E.E.E. Trans. autom. Control*, **10**, 112.

CHAO, K. S., and LIU, D. K., 1971, *I.E.E.E. Trans. autom. Control*, **16**, 100.

D'ANGELO, H., 1970, *Linear Time Varying Systems : Analysis and Synthesis* (Allyn & Bacon).

FORTMANN, T. E., 1970, *I.E.E.E. Trans. autom. Control*, **15**, 599.

GLASS, C. M., and D'ANGELO, H., 1967, *Proc. First Asilomar Conference on Circuits and Systems*, p. 172.

GRAMMATICOS, A. J., 1969, *Proc. Seventh Annual Allerton Conference on Circuit and System Theory*, p. 368.

GREVILLE, T. N. E., 1959, *SIAM Rev.*, **1**, 38.

KELLY, L. G., 1967, *Handbook of Numerical Methods and Applications* (Addison-Wesley).

MILO, S., and POLICASTRO, M., 1970, *I.E.E.E. Trans. autom. Control*, **15**, 257.

PENROSE, R., 1955, *Proc. Camb. phil. Soc.*, **1**, 406.

RAMASWAMI, B., and RAMAR, K., 1969, *I.E.E.E. Trans. autom. Control*, **14**, 417.

SARACHICK, P. E., and KREINDLER, E., 1965, *Int. J. Control*, **1**, 419.

SILVERMAN, L. M., 1966, *I.E.E.E. Trans. autom. Control*, **11**, 300 ; 1968, *Ibid.*, **13**, 436 ; 1969, *Ibid.*, **14**, 270.

SILVERMAN, L. M., and MEADOWS, H. E., 1965 a, *Proc. National Electronics Conference*, p. 689 ; 1965 b, *I.E.E.E. Trans. Circuit Theory*, **12**, 308 ; 1966, *Proc. Fourth Annual Allerton Conference on Circuit and System Theory*, p. 776 ; 1967, *J. SIAM Control*, **5**, 64.

STUBBERUD, A. R., 1963, *I.E.E.E. Trans. Applic. Ind.*, **67**, 214 ; 1964, **68**, 411.

WOLOVICH, W. A., 1968, *I.E.E.E. Trans. autom. Control*, **13**, 569.