WMMSE Design of Digital Filter Banks with Specified Composite Response

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Abstract—**A new method for designing uniform and nonuniform digital filter banks with a specified composite response is presented. The composite response of the filter bank can be met either exactly or to within a given tolerance.**

We focus on filter banks in which the individual filters are finite impulse response (FIR) digital filters of possibly nonequal length, although the new method is applicable even to more general structures as well. The new method minimizes the weighted sum of the mean square errors in the response of the individual filters, subject to the composite response specifications. Sufficient conditions for either realness or phase linearity of the optimal individual filters are presented.

The new weighted minimum mean square error (WMMSE) design method is interpreted from a statistical viewpoint as a maximization of the harmonic mean of the output signal-to-noise ratio (SNR) of the individual filters.

The complexity of the new method is analyzed, and the design process is demonstrated via a design example.

I. Introduction

In many applications, digital filter banks with a specified composite response (usually flat, or having bandpass characteristics) are required. For example, in speech recognition [1], the speech signal is analyzed by a filter bank in order to measure its time-varying energy in different frequency bands. A flat composite response guarantees that the sum of the outputs of all the individual filters restores the original input signal so that no signal component is misrepresented.

The conventional filter banks used in these applications are composed of individual filters that are finite impulse response (FIR) digital filters with linear phase and real coefficients. However, due to issues of complexity and cost-effectiveness related to the use of VLSI technologies, generalized structures of FIR filters were suggested in [3] and [4]. The proposed design method presented here is therefore derived using a generalized structure.

The well-known Remez exchange method is applicable for the design of optimal min–max FIR filters [5], and the Wiener filtering approach can be used to design optimal WMMSE FIR filters [6], [7]. In filter bank designs, these methods are used to design each filter independently of the other filters in the bank. Therefore, direct application of these methods typically results in a poor composite response [8], [10].

Various methods exist that guarantee a flat composite response [10], [12]–[14]. However, they all suffer from the following disadvantages:

1) suboptimality under both min–max and WMMSE criteria; and

2) limited flexibility in the design (e.g., restriction to individual filters of equal length, restriction to odd-length conventional FIR filters, limitations on the ratio between the passband and the stopband deviations, etc.).

Although a min–max design method of a filter bank with a specified composite response was suggested in [11], it is based on a complicated automated trial-and-error approach of iterated designs using the Remez exchange method, which is not guaranteed to converge.

In principal, one can design an optimal (min–max) filter bank subject to a specified composite response using the linear programming techniques applied in [15] for the design of a single optimal (min–max) filter. However, since typically the number of variables, which is equal to the overall number of filter coefficients in the filter bank, can well be over 1000, this design method may become quite complicated in many applications.

In this paper, we show that the WMMSE criterion can be applied to the given design problem with reasonable complexity. Furthermore, since the optimal filter bank is derived analytically, as the solution of a set of linear equations, the effect of various design parameters can be investigated. As an example, we allow a tolerance in the composite response specification and characterize the design tradeoff via a curve that relates the overall performance of the individual filters (the WMMSE) to this tolerance parameter. Moreover, using eigenvalue decomposition routines, this design curve can be derived very efficiently.

In addition, the proposed WMMSE design has the following advantages.

1) It has a statistical interpretation as the filter bank that minimizes the weighted sum of output noise powers or, equivalently, the filter bank with the maximal weighted harmonic mean of the output signal-to-noise ratios (SNR's). This interpretation is important in communication applications in which the input of the filter bank is defined statistically (e.g., detection of frequency-hopping signals).

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2) It conveniently accommodates a generalized structure for the filters in the filter bank, so that each filter can be composed of a linear combination of arbitrary basic components, with possibly a *different number* and *different types* of components in each filter. In particular, for FIR filter banks, the filters in the bank may each have a different length.

As mentioned earlier, the generalized structure is especially suitable for the use of prototype (VLSI) filters, as cheap off-the-shelf basic components [3], [4] which may be either FIR or IIR filters.

The organization of the paper is as follows.

In the next section, we present the new design method. The presentation is done for filter banks having a generalized structure and complex coefficients. Sufficient conditions on the specifications, which guarantee real coefficients and zero phase error of the individual filters, are presented in Section III. In that section, we also discuss the issue of phase linearity of conventional FIR filter banks as a particular example of the more general condition for zero phase error. Section IV is devoted to the statistical interpretation of the WMMSE criterion. This interpretation relates the filter design problem to Wiener filtering, subject to a specified composite response. In Section V, the complexity of each step in the design process is investigated, for various characterizations of the design problem. A design example of an octave-band filter bank is presented in Section VI, and conclusions are drawn in the last section. Some of the details regarding the issue of complexity of the design are given in the Appendix.

II. The WMMSE Design of Filter Banks

The formulation of the design problem is as follows.

1) The filter bank is composed of N individual digital filters. The ith filter is a linear combination of M_i basic components having frequency responses $E_{ik}(f)$ $k = 1, 2, \cdots, M_i$. This generalized structure enables us to accommodate more general building blocks as in Fig. 2 where $E_{ik}(f) = (E_i(f))^k$. For conventional FIR filters, the basic components are delays, and thus, $E_{ik}(f)$ takes the form $E_{ik}(f) = e^{-j2\pi(k+l_i)f}$ (where l_i are additional delay values, which are usually used to guarantee a linear phase composite response, as further elaborated in Section III).

All the results to be derived in this section are for complex filter banks. Thus, the coefficients of the linear combinations (denoted by a_{ik}, $k = 1, \cdots, M_i$) are assumed to be complex numbers. In the next section, we state and prove sufficient conditions for the realness of these coefficients.

The frequency response of the ith filter is, therefore,

$$H_i(f) \triangleq \sum_{k=1}^{M_i} a_{ik} E_{ik}(f). \tag{1}$$

2) The desired frequency response of the ith filter is denoted by $D_i(f)$, $i = 1, \cdots, N$. The error between this desired frequency response and the frequency response of the corresponding filter is weighted according to a specified (real) weight function $W_i(f)^2$.

We use the mean square error (MSE) as the error norm, and therefore, the ith filter response error is defined as

$$\delta_i^2 \triangleq \int_{-0.5}^{0.5} W_i(f)^2 \, |D_i(f) - H_i(f)|^2 \, df. \tag{2}$$

3) The composite response of the filter bank is the sum of the responses of the individual filters. Let the composite frequency response be denoted by $H_{N+1}(f)$. Thus, $H_{N+1}(f) = \Sigma_{i=1}^{N} H_i(f)$. The specifications on the composite response are given by a desired composite frequency response denoted by $D_{N+1}(f)$ and by a (real) weight function $W_{N+1}(f)^2$ related to the MSE norm of the composite response. Therefore, the composite response error is given by

$$\delta_{N+1}^2 = \int_{-0.5}^{0.5} W_{N+1}(f)^2 \, |D_{N+1}(f) - H_{N+1}(f)|^2 \, df. \tag{3}$$

For example, if a flat composite response is specified, $|D_{N+1}(f)| = 1$, and for equal error weighting in frequency, $W_{N+1}(f) = 1$ as well.

4) The performance of the filter bank is measured in terms of a weighted combination of the individual filter's response *errors*. The ith coefficient of this combination, denoted by K_i^2, reflects the relative importance of the ith filter specification. Thus, $K_i = 0$ means that the frequency response of the ith filter can be set arbitrarily (but subject to fulfilling the composite response specifications), whereas $K_i \to \infty$ means that the frequency response of the ith filter should be as close as possible to its desired frequency response, regardless of the composite response specifications.

The overall weighted MSE is denoted by ϵ^2 and is thus given by

$$\epsilon^2 \triangleq \sum_{i=1}^{N} K_i^2 \delta_i^2. \tag{4}$$

5) Two kinds of composite response specification are possible. The first is a tolerance specification, stated by the constraint $\delta_{N+1}^2 \leq \eta^2$, and the second is an indirect specification, by incorporating the composite response error δ_{N+1}^2 into the weighted MSE:

$$\epsilon_t^2 \triangleq \epsilon^2 + K_{N+1}^2 \delta_{N+1}^2. \tag{5}$$

6) The design problem is to find the optimal set of $M_a = \Sigma_{i=1}^{N} M_i$ coefficients $\{a_{ik}\}_{k=1,i=1}^{M_i,N}$. The optimization criterion is minimization of ϵ_t^2 or minimization of ϵ^2 subject to the composite response constraint.

Therefore, two different optimization problems can be stated:

$$\text{I.} \quad \min_{\{a_{ik}\}_{i,k}} \{\epsilon_t^2\} \tag{6a}$$

$$\text{II.} \quad \min_{\{a_{ik}\}_{i,k},\, \delta_{N+1}^2 \leq \eta^2} \{\epsilon^2\}. \tag{6b}$$

7) Both ϵ^2 and δ_{N+1}^2 are convex functions of the unknown variables. From the theory of convex programming [9, sec. 4.5], it follows immediately that the two optimization problems are equivalent. We therefore start by deriving the solution of the first optimization problem and then describe how $K_{N+1}(\eta)$ is found, in order to use this solution for the second optimization problem.

Before we derive the optimal filter bank solution, we focus on two extreme composite response specifications.

1) As $\eta \to \infty$ ($K_{N+1} \to 0$), an arbitrary composite response is allowed. Therefore, in order to minimize $\epsilon^2(\epsilon_t^2)$, we can minimize the N individual filter errors $\{\delta_i\}_{i=1}^N$ separately. The original optimization problem is thus converted into N simpler optimization problems. For conventional FIR structure, the solution of each of the N optimization problems is the Wiener filter derived in [6] and [7].

2) As $K_{N+1} \to \infty$ (η approaches its minimal possible value), the optimal composite response is obtained. If the desired composite response can be met by any filter bank of the prescribed structure (i.e., if there is at least one set of $\{a_{ik}\}_{i,k}$ for which $\delta_{N+1}^2 = 0$), then it is guaranteed that this composite response is achieved by the proposed design method. If this desired composite response is not feasible, the resulting filter bank will have a composite response that is its best possible approximation in the MSE sense.

A. Solution of the First Optimization Problem

ϵ_t^2 is clearly a p.s.d. quadratic form of the unknown variables $\{a_{ik}\}_{k=1,i=1}^{M_i,N}$. Thus, the optimal set of coefficients is given by a solution of a set of M_a linear equations. However, in most practical applications, the basic components of all the N individual filters are taken out of a set of only $M_{N+1} << M_a$ distinct elements (e.g., for conventional FIR structures, $E_{ik}(f)$ represents delays, and $M_{N+1} \triangleq \max_{i=1,\cdots,N}\{M_i\}$, which is the largest delay in the filter bank).

In this case, the size of the set of linear equations is reduced to M_{N+1}, thus reducing dramatically the complexity of the design, as elaborated further in Section V. In order to exploit this property, we introduce the following notation.

The composite frequency response $H_{N+1}(f)$ is a linear combination of the frequency responses of all the M_{N+1} distinct basic components. We order these M_{N+1} basic components arbitrarily and denote them by $E_{(N+1)k}(f)$, $k = 1, \cdots, M_{N+1}$. We denote the coefficients of the linear combination by $a_{(N+1)k}$, and thus,

$$H_{N+1}(f) = \sum_{k=1}^{M_{N+1}} a_{(N+1)k} E_{(N+1)k}(f). \tag{7}$$

Let the vector $\boldsymbol{a}_i \in \mathbb{C}^{M_i}$ represent the coefficients of the ith filter for $i = 1, \cdots, N+1$ (where $\boldsymbol{a}_{N+1}$ represents the above coefficients of the composite response).

In what follows, an augmented version of any vector $\boldsymbol{v}_i \in \mathbb{C}^{M_i}$ is a vector in $\mathbb{C}^{M_{N+1}}$, denoted by $(\boldsymbol{v}_i)^{\text{aug}}$ as defined below.

The kth element of the vector $(\boldsymbol{v}_i)^{\text{aug}}$ is zero if $E_{(N+1)k}(f)$ is not a basic component of the ith filter. Otherwise, if $E_{(N+1)k}(f) = E_{im}(f)$ for some m, $1 \le m \le M_i$, then the kth element of the vector $(\boldsymbol{v}_i)^{\text{aug}}$ is the mth element of $\boldsymbol{v}_i$.

From this definition and (7), it follows

$$\boldsymbol{a}_{N+1} = \sum_{i=1}^{N} (\boldsymbol{a}_i)^{\text{aug}}. \tag{8}$$

It is easily verified that the augmentation operation is a one-to-one mapping of $\mathbb{C}^{M_i}$ into $\mathbb{C}^{M_{N+1}}$. The reduced version of a vector $\boldsymbol{v} \in \mathbb{C}^{M_{N+1}}$, denoted by $\boldsymbol{v}]_{M_i} \in \mathbb{C}^{M_i}$, is defined as follows. If $\boldsymbol{v}$ is in the range of the augmentation operation, i.e., $\boldsymbol{v} = (\boldsymbol{v}_i)^{\text{aug}}$, then $\boldsymbol{v}]_{M_i} = \boldsymbol{v}_i$. Otherwise, the vector $\boldsymbol{v}$ is projected into the range of the augmentation by replacing the appropriate $(M_{N+1} - M_i)$ elements by zeros and is then reduced to $\mathbb{C}^{M_i}$ as defined above. Augmentation (reduction) of square matrices is done by augmenting (reducing) both the columns and the rows.

Note that augmentation (reduction) from $\mathbb{C}^{M_{N+1}}$ to itself is an identity operation; hence, in the sequel we use augmentation symbols for matrices and vectors in $\mathbb{C}^{M_{N+1}}$ as well if they are convenient for the presentation.

In the sequel, a superbar denotes complex conjugation, and $\boldsymbol{v}^H$ denotes conjugate transposition of $\boldsymbol{v}$.

Substituting (1)–(4) and (7) in (5) and rearranging the expression of ϵ_t^2 in terms of the coefficient vectors $\{\boldsymbol{a}_i\}_{i=1}^{N+1}$, we obtain the following alternative expression for the optimization problem in (6a):

$$\min_{\{\boldsymbol{a}_i\}_{i=1}^N} \sum_{i=1}^{N+1} K_i^2[(\boldsymbol{a}_i - \boldsymbol{a}_i^o)^H \mathbf{R}_i(\boldsymbol{a}_i - \boldsymbol{a}_i^o) + \hat{\delta}_i^2], \tag{9}$$

where $\boldsymbol{a}_{N+1}$ is given in (8) and $\boldsymbol{a}_i^o \triangleq \mathbf{R}_i^{-1}\boldsymbol{d}_i$ is a vector in $\mathbb{C}^{M_i}$.

The elements of the square matrix $\mathbf{R}_i$ are

$$R_i(m,k) = \int_{-0.5}^{0.5} W_i(f)^2\, \overline{E}_{im}(f)\, E_{ik}(f)\, df \qquad i = 1, \cdots, N+1; \quad m, k = 1, \cdots, M_i. \tag{10}$$

The elements of $\boldsymbol{d}_i \in \mathbb{C}^{M_i}$ are

$$d_i(m) = \int_{-0.5}^{0.5} W_i(f)^2\, \overline{E}_{im}(f)\, D_i(f)\, df \qquad i = 1, \cdots, N+1;\ m = 1, \cdots, M_i. \tag{11}$$

The value of $\hat{\delta}_i^2$ is

$$\hat{\delta}_i^2 = \int_{-0.5}^{0.5} W_i(f)^2\, |D_i(f)|^2\, df - \boldsymbol{d}_i^H \mathbf{R}_i^{-1} \boldsymbol{d}_i \qquad i = 1, \cdots, N+1. \tag{12}$$

Note that the WMMSE approach in [7] leads to the solution $\boldsymbol{a}_i = \boldsymbol{a}_i^o$, $i = 1, \cdots, N$, for which $\delta_i^2 = \hat{\delta}_i^2$ is minimal for $i \le N$. However, the optimal set of coefficients of the composite response, which is $\boldsymbol{a}_{N+1}^o$, is in general not equal to the augmented sum of these filters. Therefore, this filter bank is not necessarily the solution of (9).

The optimization problem stated in (9) is the minimization of a p.s.d quadratic from. Its analytical solution (obtained by differentiation with respect to the unknown variables) is

$$\boldsymbol{a}_i = \boldsymbol{a}_i^o + \frac{1}{K_i^2} \mathbf{R}_i^{-1}\boldsymbol{q}]_{M_i} \qquad i = 1, \cdots, N \tag{13}$$

where $\boldsymbol{q} \in \mathbb{C}^{M_{N+1}}$ is a correction vector due to the composite response specification and is given by solving the following set of linear equations:

$$\left[\sum_{i=1}^{N+1} \frac{1}{K_i^2} (\mathbf{R}_i^{-1})^{\text{aug}}\right] \boldsymbol{q} = \boldsymbol{p}. \tag{14}$$

The vector $\boldsymbol{p}$ is the difference between the optimal set of coefficients of the composite response and the augmented sum of the optimal individual filters, i.e.,

$$\boldsymbol{p} = \boldsymbol{a}_{N+1}^o - \sum_{i=1}^{N} (\boldsymbol{a}_i^o)^{\text{aug}}. \tag{15}$$

The resulting errors are

$$\delta_i^2 = \hat{\delta}_i^2 + \frac{1}{K_i^4} \boldsymbol{q}]_{M_i}^H \mathbf{R}_i^{-1}\boldsymbol{q}]_{M_i} \qquad i = 1, \cdots, N+1. \tag{16}$$

This completes the solution of the first optimization problem for weight factors that are neither zero nor approach infinity, under the assumption that the matrices $\mathbf{R}_i^{-1}$ exist and that the matrix appearing in (14) is a nonsingular matrix.

It can be shown that this assumption holds if the design problem is well defined [2]. We will now extend the results for weight values which are approaching infinity.

If $K_j \to \infty$, the jth filter desired response overrides all other specifications, thus forcing the jth filter to have the minimal error $\hat{\delta}_j^2$ (i.e., in (13) we get in the limit, as $K_j \to \infty$, that $\boldsymbol{a}_j = \boldsymbol{a}_j^o$). This affects the correction vector $\boldsymbol{q}$ by omitting $(\mathbf{R}_j^{-1})^{\text{aug}}/K_j^2$ from (14) since this value approaches zero as $K_j \to \infty$. However, as shown in the sequel, increasing the value of K_j results in an increase of the overall error of the remaining filters.

In particular, $K_{N+1}^2 \to \infty$ corresponds to a constraint on the composite response, and in this case, $\mathbf{R}_{N+1}^{-1}/K_{N+1}^2$ is omitted in (14), thus increasing the overall error of the individual filters. The solution then coincides with an earlier result we presented in [8]. As mentioned earlier, the composite response error δ_{N+1}^2 is minimized in that case.

The basic algorithm can be modified to accommodate the case of zero weight values. Since this modification is lengthy and of less importance, it is not presented here and can be found in [2].

B. Solution of the Second Optimization Problem

The second optimization problem, stated in (6b), can be solved by converting it to the problem in (6a) which we just solved. This is done by finding the weighting factor K_{N+1}, which incorporates the composite response specification into (6a), from the given tolerance η on the composite response error. We describe now an algorithm for computing $K_{N+1}(\eta)$.

Rewriting (14), we obtain the following relation between $\boldsymbol{q}$ and the value of K_{N+1}:

$$\left(\frac{1}{K_{N+1}^2} \mathbf{R}_{N+1}^{-1} + \mathbf{T}\right) \boldsymbol{q} = \boldsymbol{p} \tag{17}$$

where $\mathbf{T} \triangleq \Sigma_{i=1}^N (1/K_i^2) (\mathbf{R}_i^{-1})^{\text{aug}}$ is an $M_{N+1} \times M_{N+1}$ matrix which is independent of K_{N+1}. Equations (16) and (17) give an implicit relation between the values of δ_{N+1}^2 and K_{N+1}. In order to find an explicit relation, we make a change of basis in (17) so that both $\mathbf{R}_{N+1}^{-1}$ and $\mathbf{T}$ become diagonal in the new basis of $\mathbb{C}^{M_{N+1}}$. For that purpose, we use the following lemma, which is easily derived from [17, theorem 7.12.2].

Lemma 1: For any two Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, with $\mathbf{A}$ being a p.d. matrix, there exists a nonsingular matrix $\mathbf{V}$ such that $\mathbf{V}^H\mathbf{A}\mathbf{V} = \mathbf{I}$ and $\mathbf{V}^H\mathbf{B}\mathbf{V} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix.

The matrix $\mathbf{R}_{N+1}$ is clearly a p.s.d. Hermitian matrix, and for well-defined design problems, it is nonsingular. Thus, $\mathbf{R}_{N+1}^{-1}$ exists and is a p.d. Hermitian matrix. The matrix $\mathbf{T}$ is also a p.s.d. Hermitian matrix (being an augmented sum of p.d. Hermitian matrices $\mathbf{R}_i^{-1}$). Thus, lemma 1 holds for the pair of matrices $\mathbf{R}_{N+1}^{-1}$ and $\mathbf{T}$, and there is a nonsingular matrix $\mathbf{V}$ of dimension $M_{N+1} \times M_{N+1}$, so that

$$\mathbf{V}^H\mathbf{R}_{N+1}^{-1}\mathbf{V} = \mathbf{I} \tag{18a}$$

$$\mathbf{V}^H\mathbf{T}\mathbf{V} = \text{diag}\,(d_1, \cdots, d_{M_{N+1}}). \tag{18b}$$

We can express d_n in terms of $\boldsymbol{v}_n$, the nth column of $\mathbf{V}$, as follows: $d_n = \boldsymbol{v}_n^H\mathbf{T}\boldsymbol{v}_n$. Since $\mathbf{T}$ is p.s.d., this expression implies that $d_n \geq 0$. The matrix $\mathbf{V}$ is nonsingular, and therefore, we can perform a change of variables from $\boldsymbol{p}$, $\boldsymbol{q}$ to $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{q}}$ as follows:

$$\boldsymbol{q} = \mathbf{V}\hat{\boldsymbol{q}} \tag{19a}$$

$$\hat{\boldsymbol{p}} = \mathbf{V}^H\boldsymbol{p}. \tag{19b}$$

Multiplying (17) by $\mathbf{V}^H$ and using (18) and (19), we obtain M_{N+1} scalar equations:

$$\hat{q}_n \triangleq \frac{K_{N+1}^2\hat{p}_n}{1 + K_{N+1}^2 d_n} \qquad n = 1, \cdots, M_{N+1}. \tag{20}$$

Substituting (20) and (19a) in (16), we obtain the following relation between K_{N+1}^2 and δ_{N+1}^2:

$$\delta_{N+1}^2 = \hat{\delta}_{N+1}^2 + \sum_{n=1}^{M_{N+1}} \frac{|\hat{p}_n|^2}{(1 + K_{N+1}^2 d_n)^2}. \tag{21}$$

From (16) and the definition of the filter bank error ϵ^2 in (4) we obtain

$$\epsilon^2 = \hat{\epsilon}^2 + \boldsymbol{q}^H\mathbf{T}\boldsymbol{q} \tag{22}$$

where $\hat{\epsilon}^2 = \Sigma_{i=1}^N \hat{\delta}_i^2$ is the error of the optimal filter bank with unspecified composite response ($\eta \to \infty$). Substituting (19a), (18b), and (20) in (22), we obtain

$$\epsilon^2 = \hat{\epsilon}^2 + \sum_{n=1}^{M_{N+1}} d_n \left(\frac{K_{N+1}^2 |\hat{p}_n|}{1 + K_{N+1}^2 d_n} \right)^2. \tag{23}$$

Using (18) and (19b), we can evaluate the values of $\lim_{K_{N+1}^2 \to 0} (\delta_{N+1}^2)$ and $\lim_{K_{N+1}^2 \to \infty} (\epsilon^2)$ from (21) and (23) and obtain

$$\tilde{\delta}_{N+1}^2 \triangleq \lim_{K_{N+1}^2 \to 0} (\delta_{N+1}^2) = \hat{\delta}_{N+1}^2 + p^H \mathbf{R}_{N+1} p \tag{24a}$$

$$\tilde{\epsilon}^2 \triangleq \lim_{K_{N+1}^2 \to \infty} (\epsilon^2) = \hat{\epsilon}^2 + p^H \mathbf{T}^{-1} p. \tag{24b}$$

It is easily verified from (21) and (23) that δ_{N+1}^2 is a monotonically decreasing function of K_{N+1}^2, and ϵ^2 is a monotonically increasing function of K_{N+1}^2. Upper and lower limits of these two functions are $\tilde{\delta}_{N+1}^2(\tilde{\epsilon}^2)$ and $\hat{\delta}_{N+1}^2(\hat{\epsilon}^2)$, respectively. The following important property is obtained by evaluating $d(\delta_{N+1}^2)/d(K_{N+1}^2)$ and $d(\epsilon^2)/d(K_{N+1}^2)$ from (21) and (23), respectively:

$$\frac{d(\epsilon^2)}{d(\delta_{N+1}^2)} = -K_{N+1}^2. \tag{25}$$

From (25) it follows that the design curve of ϵ^2 as function of δ_{N+1}^2 is a monotonically decreasing convex curve, and K_{N+1}^2 has a geometric interpretation as the slope of the curve at the point $(\delta_{N+1}^2, \epsilon^2)$. Fig. 1(a) illustrates a typical design curve.

Three different ranges of the value of the tolerance specification η^2 should now be considered.

The first range is $\eta^2 < \hat{\delta}_{N+1}^2$, in which the tolerance specification is actually irrelevant since there exists no filter bank of the given structure that can fulfill this specification. Values in the second range, defined by $\eta^2 \geq \tilde{\delta}_{N+1}^2$, are actually fulfilled by the optimal filter bank, which ignores the composite response specifications (the Wiener solution a_i^o). The third range is $\tilde{\delta}_{N+1}^2 > \eta^2 \geq \hat{\delta}_{N+1}^2$, and in this case, since ϵ^2 is a monotonically decreasing function of δ_{N+1}^2, $\delta_{N+1}^2 = \eta^2$. Thus, for the latter situation, we have to solve the nonlinear scalar equation derived from (21), namely, $\eta^2 = \delta_{N+1}^2(K_{N+1}^2)$, in order to evaluate K_{N+1}^2. An alternative approach is to first draw the design curve $\epsilon^2(\delta_{N+1}^2)$ using (21) and (23), then choose the desired point on this curve, and find K_{N+1}^2 geometrically, as illustrated in Fig. 1(b).

Furthermore, upper and lower bounds on $K_{(N+1)}^2(\eta^2)$, which reduce the number of iterations needed in evaluating $K_{(N+1)}^2(\eta^2)$ by numerical methods, are derived in [2].

Note: The design curve of $(\epsilon_t^2 - K_j^2 \delta_j^2)$ as a function of K_j^2 has the same properties as the design curve of ϵ^2 as a function of K_{N+1}^2. Therefore, our remark that an increase in the value of K_j^2 implies an increase in the overall error of the remaining filters (i.e., $\epsilon_t^2 - K_j^2 \delta_j^2$) follows as a consequence of (23).

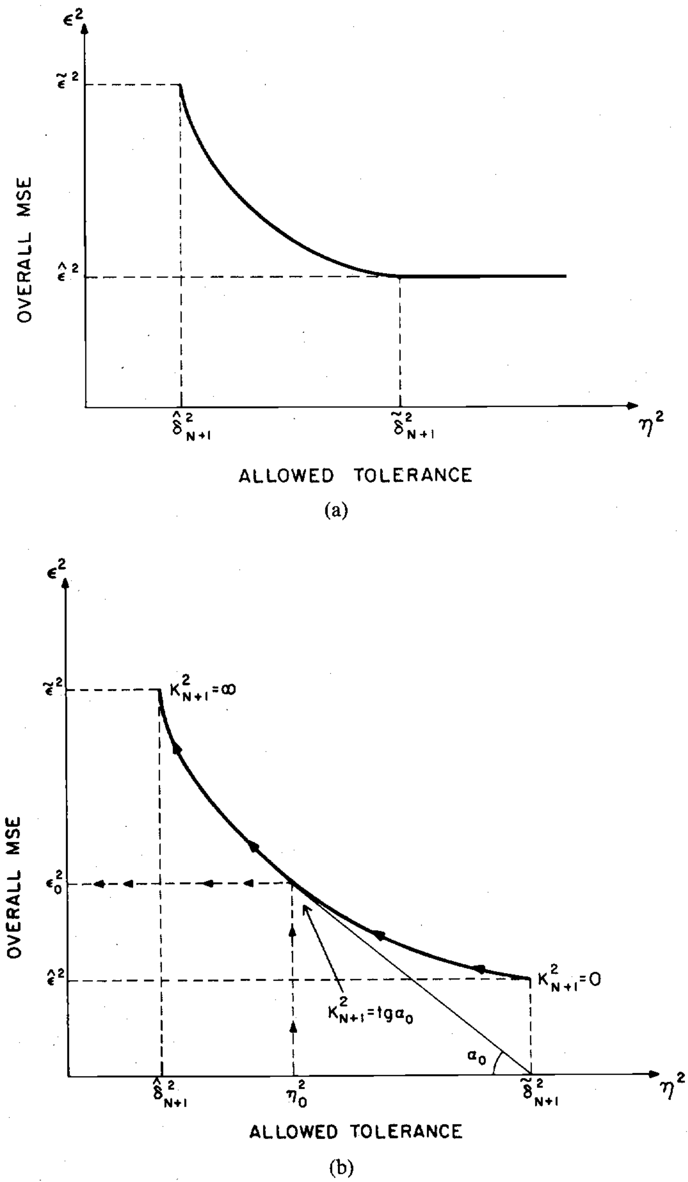


Fig. 1. (a) Typical tradeoff curve of the MSE ϵ^2 as function of the allowed tolerance from the specified composite response (η^2). (b) Geometrical interpretation of the weight constant K_{N+1}^2.

III. Phase Linearity and Realness of Optimal Filter Banks

In the general model presented in the previous section, we assumed that the designed filter bank has complex coefficients $\{a_{ik}\}_{i=1,k=1}^{N,M_i}$, and we have not considered the issue of phase linearity of the resulting filters. We discuss the subjects of realness and phase linearity in this section.

Two theorems are presented. The first provides a sufficient condition for realness of the optimal filter bank coefficients, and the second provides a sufficient condition for zero phase error in the responses of all N individual filters in the bank. These two theorems are derived for the general structure defined in Section II. However, their interpretation for the important class of FIR filter banks is given by means of corollaries following the relevant theorem. Theorem 1 provides sufficient conditions for the realness of the coefficients of the optimal filter bank.

Theorem 1: If a) $W_i(f)^2 = W_i(-f)^2$, $i = 1, \cdots, N + 1$, and a function $\vartheta(f)$ exists such that the following conditions are satisfied:

b) $D_i(f) = \overline{D}_i(-f)\, e^{j\vartheta(f)} \qquad i = 1, \cdots, N + 1$

c) $E_{im}(f) = \overline{E}_{im}(-f)\, e^{j\vartheta(f)} \qquad i = 1, \cdots, N + 1; \quad m = 1, \cdots, M_i,$

then all the filters in the *optimal* filter bank have real coefficients. Furthermore, the matrices $\{\mathbf{R}_i\}_{i=1}^{N+1}$ and the vectors $\{\boldsymbol{d}_i, \boldsymbol{a}_i^o\}_{i=1}^{N+1}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ are all real. Thus, the design process then involves only operations on real numbers.

Proof: It is easily verified from (10) and (11) that conditions a)–c) are sufficient conditions for realness of the matrices $\mathbf{R}_i$ and the vectors $\boldsymbol{d}_i$. Since $\{\boldsymbol{a}_i^o\}_{i=1}^{N+1}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ are given in terms of these values, they are all real vectors, and so are the coefficients of the individual filters of the optimal filter bank.

Corollary 1: For a filter bank composed of conventional FIR filters (i.e., $E_{ik}(f) = e^{-j2\pi f(k+l_i)}$), the individual filters in the optimal filter bank have real coefficients, provided that all the impulse responses related to the desired frequency responses $D_i(f)$ and weight functions $W_i(f)^2$ are real sequences.

Proof: For conventional FIR filters, condition c) is satisfied with $\vartheta(f) = 0$. The corollary thus restates conditions a) and b) for $\vartheta(f) = 0$, in a slightly different manner.

Theorem 2 below provides sufficient conditions for exact fulfillment of the desired phase response specifications (up to an integer multiple of π due to sign inversions).

Theorem 2: Under the following two conditions:

a) all the filters in the bank have the *same* desired phase response (up to an integer multiple of π) $\psi(f)$, i.e.,

$$D_i(f) = \hat{D}_i(f)\, e^{j\psi(f)} \qquad i = 1, \cdots, N + 1,$$

with $\hat{D}_i(f)$ being real functions; and

b) the basic components of each filter can be divided into distinct pairs such that in every pair the frequency response of one component is the complex conjugate of the frequency response of the other component multipled by $e^{j2\psi(f)}$; written formally, for every i, $i = 1, \cdots, N + 1$, there exists a permutation π_i such that $(\forall k)\ (E_{ik}(f) = \overline{E}_{i\pi_i(k)}(f)\, e^{j2\psi(f)})$.

The phase response of each filter in the optimal filter bank is exactly the desired phase response (up to an integer multiple of π), i.e., $H_i(f) = \hat{H}_i(f)\, e^{j\psi(f)}$, $i = 1, \cdots, N + 1$, with $\hat{H}_i(f)$ being real functions.

Proof: It is easily verified that conditions a) and b) are sufficient for the following results:

1) $d_i(m) = \overline{d}_i(\pi_i(m))$ for all m and i; and
2) $R_i(m, k) = \overline{R}_i(\pi_i(m), \pi_i(k))$ for all m, k, and i.

It is easily proven that from 2) follows

3) $R_i^{-1}(m, k) = \overline{R_i^{-1}}(\pi_i(m), \pi_i(k))$ for all m, k, and i.

From 1) and 3) it follows directly that

4) $a_i^o(m) = \overline{a}_i^o(\pi_i(m))$ for all m and i.

Since 3) and 4) hold for $i = N + 1$, the augmentation of $\boldsymbol{a}_i^o$ and $\mathbf{R}_i^{-1}$ only reorders the pairs of elements according to $\pi_{N+1}(\cdot)$ instead of $\pi_i(\cdot)$, and therefore,

5) $p(m) = \overline{p}(\pi_{N+1}(m))$ for all m; and
6) $T(m, k) = \overline{T}(\pi_{N+1}(m), \pi_{N+1}(k))$ for all m, k.

Thus,

7) $q(m) = \overline{q}(\pi_{N+1}(m))$ for all m.

The reduction of $\boldsymbol{q}$ to $\boldsymbol{q}]_{M_i}$ reorders the pairs of elements according to the permutation $\pi_i(\cdot)$, and therefore,

8) $q]_{M_i}(m) = \overline{q}]_{M_i}(\pi_i(m))$ for all m and i.

And the final result from 3), 4), and 8) is that the optimal coefficients satisfy

9) $a_i(m) = \overline{a}_i(\pi_i(m))$ for all m and i.

Combining 9) above with condition b), one easily obtains the result that $H_i(f) = \overline{H}_i(f)\, e^{j2\psi(f)}$, which means that $H_i(f) = \hat{H}_i(f)\, e^{j\psi(f)}$, with $\hat{H}_i(f)$ being real functions.

Note: Similar results hold for $\hat{D}_i(f)$ being pure imaginary functions, with $\hat{H}_i(f)$ being pure imaginary, and a $(-)$ sign in 1)–9).

Corollary 2: For filter banks composed of conventional FIR filters (i.e., $E_{ik}(f) = e^{-j2\pi f(k+l_i)}$), with the additional delay values being $l_i = \frac{1}{2}(M_{N+1} - M_i) - 1$, so that all N filters have the same delay, and desired frequency responses $D_i(f)$, which have the same linear phase response $\psi(f) = -\pi f(M_{N+1} - 1)$, the optimal filters are also linear phase filters.

Proof: For $l_i = \frac{1}{2}(M_{N+1} - M_i) - 1$ and $\psi(f) = -\pi f(M_{N+1} - 1)$, it is easily verified that condition b) of theorem 2 holds for $\pi_i(n) = (M_i + 1 - n)$. Condition a) was satisfied in the corollary statement, and thus the result follows from theorem 2.

It should be noted that the phase linearity of the optimal filter bank is not obtained when odd-length and even-length filters are mixed together in the same filter bank since then some of the additional delays of the individual filters (l_i values) involve half-sample delays, which are difficult to realize.

IV. Statistical Interpretation of the WMMSE Method

The statistical interpretation of the WMMSE criterion for the design of a single FIR filter was presented in [6]. We present here its extension to the design of filter banks, composed of FIR filters, with a specified composite response. This interpretation is useful for applications in which the input process has a statistical characterization.

Since each filter in the filter bank is usually designed to pass a different frequency band of the common input signal, we may define differently the so-called signal and noise components for each filter in the bank. The convention taken here is to consider all the frequency components of the common input which are in the passband of the ith filter as its input signal s_i and all the components in the stopband as noise n_i. Because we deal with a filter design problem, components in the transition bands of each individual filter are ignored. Thus, we view each filter as having its own input, denoted by $x_i = s_i + n_i$, for the ith filter in the bank. Note that according to the above convention, the inputs x_i, $i = 1, \cdots, N$, are not identical, unless all the transition bands are eliminated (i.e., set to have zero bandwidth). For the mathematical development, it is convenient to apply the following vector notation.

The impulse response of the ith filter, which is of length M_i, is denoted by the vector $\boldsymbol{a}_i$. The input vector, which comprises of M_i consecutive samples of the random process x_i, is denoted by $X_i(k)$, i.e., $X_i(k) = [x_i(k), \cdots, x_i(k-(M_i-1))]^T$. Thus, the corresponding output is $y_i(k) = \boldsymbol{a}_i^T X_i(k)$. As explained above, we regard input samples as being the sum of signal samples and noise samples, and we assume that they are samples of two wide-sense stationary continuous random processes. The *desired* signal at the ith output at time k is defined to be the delayed version of the input signal, i.e., $y_i^d(k) = s_i(k - \rho_i)$. We divert now from the usual convention of assuming that the signal component at the output is the response of the filter to the signal component at the input and instead set the signal component at the ith filter *output* to be the desired response $y_i^d(k)$, which is independent of the filter $\boldsymbol{a}_i$. This way, the noise component at the output of the ith filter contains both the filtered input noise and the distortions of the input signal introduced by the ith filter. With these assumptions, the signal power at the output of the ith filter is given by

$$\mathbf{S}_{oi} = E[|y_i^d(k)|^2] = \sigma_{si}^2, \tag{26}$$

and the corresponding noise power is

$$\begin{aligned}\mathbf{N}_{oi} &= E[|y_i(k) - y_i^d(k)|^2] \\ &= (\boldsymbol{a}_i - \boldsymbol{a}_i^o)^H \mathbf{R}_i(\boldsymbol{a}_i - \boldsymbol{a}_i^o) + \hat{\delta}_i^2\end{aligned} \tag{27}$$

where

$$\boldsymbol{a}_i^o \triangleq \mathbf{R}_i^{-1}\boldsymbol{d}_i \tag{28}$$

and

$$\hat{\delta}_i^2 \triangleq \sigma_{si}^2 - \boldsymbol{d}_i^H\mathbf{R}_i^{-1}\boldsymbol{d}_i. \tag{29}$$

$\mathbf{R}_i$ is an $M_i \times M_i$ autocorrelation (Toeplitz) matrix defined by

$$\mathbf{R}_i \triangleq E[\overline{X}_i(k)\, X_i(k)^T], \tag{30}$$

with $\boldsymbol{d}_i \in \mathbb{C}^{M_i}$ defined by

$$\boldsymbol{d}_i \triangleq E[s_i(k - \rho_i)\, \overline{X}_i(k)]. \tag{31}$$

$\boldsymbol{a}_i^o$ is exactly the Wiener filter coefficient vector which minimizes the output noise power of the ith filter [6]. This filter also maximizes the output SNR of the ith filter since $\mathbf{S}_{oi}$ is independent of the filter $\boldsymbol{a}_i$. Independent designs of the individual filters, using the Wiener filters, may result, however, in a poor composite response. To solve this problem, a composite response specification is now incorporated into the design process. The desired composite response is specified as the frequency response of a desired FIR filter $\boldsymbol{a}_{N+1}^o$ (e.g., $\boldsymbol{a}_{N+1}^o$, which is a unit vector, represents a flat composite response). Since each filter has a different length and delay, the composite response of the filter bank is an augmented sum of the coefficients of the individual filters, i.e., $\boldsymbol{a}_{N+1} \triangleq \Sigma_{i=1}^N (\boldsymbol{a}_i)^{\text{aug}}$. The augmentation operation takes care of the different lengths as well as the additional delays needed. The composite response error measure is a weighted MSE between the desired response $\boldsymbol{a}_{N+1}^o$ and the actual response $\boldsymbol{a}_{N+1}$, i.e., the composite response "noise" is

$$\mathbf{N}_{o(N+1)} = (\boldsymbol{a}_{N+1} - \boldsymbol{a}_{N+1}^o)^H \mathbf{R}_{N+1} (\boldsymbol{a}_{N+1} - \boldsymbol{a}_{N+1}^o) \tag{32}$$

where $\mathbf{R}_{N+1}$ is an $M_{N+1} \times M_{N+1}$ p.d. matrix [with $M_{N+1} = \max_i (M_i)$]. This specific error measure is used in order to obtain a statistical interpretation to the WMMSE criteria. If $\mathbf{R}_{N+1}$ is a Toeplitz matrix, one can interpret $\mathbf{N}_{o(N+1)}$ as the weighted L_2 norm of the composite frequency response error. Using Parseval's theorem, the frequency weight function is given by

$$W_{N+1}(f)^2 = F\{R_{N+1}(k + d, k)\} \tag{33}$$

with $F\{\cdot\}$ representing the Fourier transform of the autocorrelation sequence with respect to the variable d.

The optimal filter bank is the filter bank with the minimal weighted sum of output noise powers, among all the filter banks having composite response noise power which is below η^2. Written formally,

$$\min_{\{a_i\}_{i=1}^N,\, \mathbf{N}_{o(N+1)} \le \eta^2} \sum_{i=1}^N K_i^2((\boldsymbol{a}_i - \boldsymbol{a}_i^o)^H \mathbf{R}_i(\boldsymbol{a}_i - \boldsymbol{a}_i^o) + \hat{\delta}_i^2). \tag{34}$$

This is exactly the second optimization problem presented in Section II, for the special case of FIR filters (see (9) for comparison).

Since the output signal powers of the individual filters are independent of the coefficients of the filters, it follows that the above-defined optimal filter bank also maximizes the weighted harmonic mean of the output SNR's, i.e., it is the solution of

$$\max_{\{a_i\}_{i=1}^N;\, \mathbf{N}_{o(N+1)} \le \eta^2} \left\{ \frac{1}{\frac{1}{N}\sum_{i=1}^N C_i \left(\frac{\mathbf{N}_{oi}}{\mathbf{S}_{oi}}\right)} \right\} \tag{35}$$

where $C_i \triangleq \sigma_{si}^2 K_i^2$.

We have thus presented the equivalence in the filter bank design problem between minimal-noise powers, maximal output SNR's, and WMMSE criteria. Furthermore, we can relate the desired frequency responses and the weighting functions to signal and noise spectra by comparing the statistical and deterministic definitions of

$\mathbf{R}_i$ and $\boldsymbol{d}_i$. This is done under the assumption that $\rho_i = (1/2)(M_i - 1)$ and $l_i = (1/2)(M_{N+1} - M_i) - 1$, and therefore, corollary 2 from the previous section holds, and all the individual filters have linear phase. It follows that each weighting function represents the spectrum of the corresponding input and each desired frequency response is the cross spectra of the corresponding input and its signal component divided by the spectrum of the input. Written formally,

$$W_i(f)^2 = F\{E[\bar{x}_i(k)\, x_i(k + d)]\} \qquad i = 1, \cdots, N \tag{36}$$

$$W_i(f)^2|D_i(f)| = F\{E[\bar{x}_i(k)\, s_i(k + d)]\} \qquad i = 1, \cdots, N. \tag{37}$$

In communication applications, the input process is characterized by its autocorrelation sequence and its cross correlation with the desired signal. Equations (36) and (37) enable us to use the new design method for these applications by suggesting a way of choosing the weight functions and desired frequency responses in terms of the autocorrelation and cross-correlation sequences. Furthermore, subject to these relations, (35) gives an interpretation of the design criteria in terms of the output SNR's.

V. On the Complexity of the WMMSE Design Method

The design of an optimal filter bank is composed of the following steps.

1) Evaluation of $\mathbf{R}_i$ and $\boldsymbol{d}_i$ from the specified frequency responses [according to (10) and (11)].

2) Calculation of $\mathbf{R}_i^{-1}$ and $\boldsymbol{a}_i^o$ (to obtain the filter bank for unspecified composite response).

3) Computation of $\boldsymbol{p}$ where if either $\boldsymbol{p} = \mathbf{0}$ or $K_{N+1} = 0$ the design is complete.

4) For specified values of $K_{N+1} > 0$, find $\boldsymbol{q}$ by solving (14).

5) For specified values of $\eta > 0$,

a) find the matrix $\mathbf{V}$ and the values of $\{d_1 \cdots d_{M_{N+1}}\}$ defined in (18);

b) compute $\hat{\boldsymbol{p}}$ and solve the nonlinear scalar equation (21) for K_{N+1}^2; and

c) given the value of K_{N+1}^2, $\boldsymbol{q}$ is computed via (20) and (19a).

6) Once $\boldsymbol{q}$ is known, the filters' coefficients are obtained by (13).

We analyze now the complexity of each of the above steps.

Step 1): In general, there are $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i^2)$ integrals to be evaluated in this step (where $\mathbf{O}(\cdot)$ denotes "order of"). It is highly complicated to evaluate these integrals numerically. However, for weighting functions that are piecewise linear, and basic components that are FIR filters, the integrals that define the matrices $\mathbf{R}_i$ can be evaluated analytically.

Let B_i be the number of distinct pieces in the ith weighting function; then the $\mathbf{R}_i$ matrices can be evaluated in $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i^2 B_i)$ operations. The desired frequency responses are present only in $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i)$ integrals, and thus, the complexity of step 1) is unaffected whether or not these responses are piecewise linear. For the special case of conventional FIR filters, and piecewise linear desired responses, the matrices $\mathbf{R}_i$ are Toeplitz matrices; thus, only M_i elements have to be evaluated for each matrix, and the integrals involving the desired responses can be evaluated analytically. Therefore, the overall complexity is $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i \hat{B}_i)$ where $\hat{B}_i$ is the number of distinct pieces in $W_i(f)^2 D_i(f)$.

Step 2): This step involves solving $(N + 1)$ systems of linear equations or, alternatively, calculating $(N + 1)$ inverses of the matrices $\mathbf{R}_i$. The complexity of this step is thus $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i^3)$. For filters composed of all-pass sections, as illustrated in Fig. 2(a), the matrices $\mathbf{R}_i$ are Toeplitz matrices. Similarly, for filters composed of sections having the same phase response and powers of a basic magnitude response, as illutrated in Fig. 2(b), the matrices $\mathbf{R}_i$ are Hankel matrices.[1] The first structure coincides with the conventional FIR structure for $\Phi_i(f) = -2\pi f$. Furthermore, this structure seems suitable for the design of filter banks composed of IIR filters. In this case, $C_i(f)$ represents an IIR filter designed so that its magnitude response is very close to the desired magnitude response of the ith filter, and the all-pass sections are used for the phase correction needed to approximate the desired (linear) phase response. The second structure is especially suitable for the design of filters based on short FIR filters in cascade. In that case, $C_i(f) = e^{-j2\pi f\rho_i}$ is the delay that guarantees causality of the ith filter, and $|A_i(f)|$ is the magnitude response of the short prototype FIR filter. This is exactly the structure used in [3], [4]. For both structures, the matrices $\mathbf{R}_i$ are invertible in $\mathbf{O}(M_i^2)$ operations, and the overall complexity of step 2) reduces to $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i^2)$.

Step 3): This step is of negligible complexity ($\mathbf{O}(\Sigma_{i=1}^{N+1} M_i)$).

Step 4): This step involves the solution of a set of M_{N+1} linear equations, and its complexity is thus $\mathbf{O}(M_{N+1}^3)$. This step is of negligible complexity in comparison to step 2) for the general case, but it dominates the complexity of the design when we deal with conventional FIR filters since the equation matrix in step 4) is not a Toeplitz matrix while all the $\mathbf{R}_i$ matrices are Toeplitz matrices.

Step 5): This step involves three different operations, out of which the third (reconstructing the vector $\boldsymbol{q}$) is of negligible complexity ($\mathbf{O}(M_{N+1}^2)$) compared to step 2). The second operation involves the solution of a nonlinear scalar equation, and its complexity can be estimated by $\mathbf{O}(M_{N+1} N_{\text{iter}})$ where N_{iter} is the number of iterations in the solution (i.e., number of values of K_{N+1}^2 tried until convergence). Since $M_{N+1} \gg 1$, this complexity can be regarded as negligible in comparison to the other design

[1] $R_i(m, k) = \int_{-0.5}^{0.5} W_i(f)^2 |C_i(f)|^2 |A_i(f)|^{m+k} df$ gives a Hankel matrix for the structure in Fig. 2(b), and $R_i(m, k) = \int_{-0.5}^{0.5} W_i(f)^2 |C_i(f)|^2 e^{-j\Phi_i(f)(m-k)} df$ gives a Toeplitz matrix for the structure of Fig. 2(a).

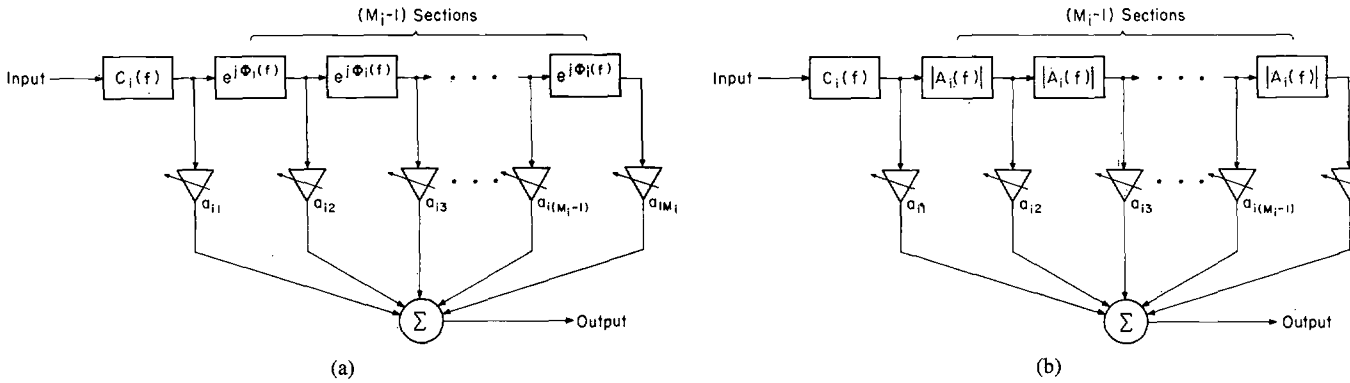


Fig. 2. (a) The structure of the ith filter for which $\mathbf{R}_i$ is a Toeplitz matrix. (b) The structure of the ith filter for which $\mathbf{R}_i$ is a Hankel matrix.

TABLE I
COMPLEXITY ANALYSIS

Problem Characteristics	Complexity	Dominant Step
General weighting functions or non-FIR basic components	$\mathbf{O}\left(\sum_{i=1}^{N+1} M_i^2\right)^a$	1)
Piecewise linear weighting functions and FIR basic components	$\mathbf{O}\left(\sum_{i=1}^{N+1} M_i^3 + \sum_{i=1}^{N+1} M_i^2 B_i\right)$	1) and 2)
Conventional FIR filters, or specific structures (Fig. 2)	$\mathbf{O}\left(M_{N+1}^3 + \sum_{i=1}^{N+1} M_i^2\right)$	2) and 4) or 5a)
Conventional FIR filters, no composite response specification	$\mathbf{O}\left(\sum_{i=1}^{N} M_i^2\right)$	2)

[a] Each operation in this row is a numerical integration.

steps. Thus, the dominant operation in step 5) is the evaluation of the matrix $\mathbf{V}$, and its complexity is about $\mathbf{O}(M_{N+1}^3)$, as discussed in the Appendix.

Step 6): This step, which concludes the design process, involves matrix vector multiplications, and therefore its complexity is $\mathbf{O}(\Sigma_{i=1}^{N+1} M_i^2)$.

Table I summarizes the overall complexity of the design procedure. Note that no distinction is made between the two types of composite response specification since the complexity of steps 4) and 5) is about the same.

VI. Design Example

To illustrate the new method, the following design example is presented.[2]

The problem we consider is the design of an octave-band filter bank composed of five filters. The composite response is specified to be flat. The first filter in the bank is a low-pass filter, the last one is a high-pass filter, and the other three are bandpass filters. The ith filter has a bandwidth which is twice the bandwidth of the (i-1)th filter (except for the first two filters, which have the same bandwidth), starting with a low-pass filter having a passband width of 200 Hz. The transition bandwidths of the ith filter are proportional to its bandwidth. Thus, the last (highest) filter has the widest transition band. The individual filters are conventional FIR filters, and the sampling frequency is 8000 Hz. In order that all the filters have the same performance, the product "filter length times transition bandwidth" is set to be about the same for all five filters [16]. For this reason, all the weight factors K_i are equal ($K_i = 1$), except for the composite response factor K_6 that takes different values for different designs. For real-time applications, an upper bound of 1.68×10^6 multiplies per second is allowed in a particular implementation of the filter bank. This leads to filter lengths ranging from 19 to 139 samples (taking advantage of the linear phase). The desired responses $D_i(f)$, $i = 1, \cdots, 5$, are the responses of ideal low-pass/bandpass/high-pass filters, respectively, i.e., set to one in the desired passband and zero elsewhere. Table II summarizes the exact passband/stopband frequencies of the individual filters and their lengths. The magnitude of the desired composite response $D_6(f)$ is unity. All six frequency responses $D_i(f)$, $i = 1, \cdots, 6$, have the same linear phase $\psi(f) = -\pi f\, 138$, which corresponds to a delay of 69

[2] The Fortran program used (run on an HP-1000 computer) can be obtained from the authors upon request.

TABLE II
DESIGN SPECIFICATIONS

Filter No. and Type	Filter Length	Lower Stopband (Hz) $D = 0,\ W^2 = 4$	Passband (Hz) $D = 1,\ W^2 = 1$	Higher Stopband (Hz) $D = 0,\ W^2 = 9$	Error in Unconstrained Design ($\hat{\delta}_i$)	Error in Flat Composite Design ($\hat{\delta}_i$)
1(LPF)	139	—	0000–0200	0300–4000	0.504×10^{-2}	1.914×10^{-2}
2(BPF)	139	0000–0200	0300–0500	0600–4000	0.849×10^{-2}	2.518×10^{-2}
3(BPF)	79	0000–0400	0600–1000	1200–4000	0.553×10^{-2}	2.881×10^{-2}
4(BPF)	39	0000–0800	1200–2000	2400–4000	0.816×10^{-2}	3.520×10^{-2}
5(HPF)	19	0000–1600	2400–4000	—	0.746×10^{-2}	2.396×10^{-2}
6(Composite response)	139	—	0000–4000	—	15.582×10^{-2}	0
rmse ϵ	—	—	—	—	1.582×10^{-2}	6.036×10^{-2}

samples. Additional delays of $69 - (1/2)(M_i - 1)$ samples are required so that all five filters have the same delay, and thus, the conditions of corollary 2 are satisfied and the resulting filters have linear phase.

The weight functions $W_i(f)^2$ for $i = 1, \cdots, 6$ are all piecewise constant functions. For each filter in the bank, the weight function $W_i(f)^2$ equals 1 in the passband, 0 in the transition bands, 4 in the lower stopband, and 9 in the upper stopband. For the composite response, a unity weight function is used; thus, δ_6^2 is the energy of the composite response error. Since $W_i(f)^2$ and $D_i(f)$, $i = 1, \cdots, 6$, correspond to real (possibly infinite) sequences, the conditions of corollary 1 are satisfied, and the optimal individual filters are all real-valued FIR filters. Two extremal values of K_6^2 are used: $K_6^2 = 0$ and $K_6^2 \to \infty$. For the first case, the resulting filter bank is composed of optimal filters that can be designed either by the new method or by the design method in [6] and [7] since the composite response is of no relevance. This design obtains the minimal weighted MSE for each individual filter $\hat{\delta}_i^2$ and the minimal MSE $\hat{\epsilon}^2$. However, since the composite response is ignored in this design, the result is a very poor response, as illustrated in Fig. 3 by the solid line. The second extremal case is obtained by specifying a flat composite response as a design constraint. This leads to a filter bank with a flat composite response (demonstrated by the dashed line of Fig. 3), and the new method minimizes the overall MSE subject to this constraint. The optimal filters obtained using $K_6^2 \to \infty$ are, of course, degraded with respect to those obtained using $K_6^2 = 0$, and their MSE is $\tilde{\epsilon}^2$, i.e., a worse performance than for any finite value of K_6^2. On the other hand, these filters minimize the composite response error, which is zero here (i.e., $\delta_6 = \hat{\delta}_6 = 0$), whereas the filters obtained using $K_6^2 = 0$ have a composite response error of $\delta_6 = \tilde{\delta}_6 = 0.156$. A tradeoff between these extremal cases can be obtained either by using finite values of K_6^2 or by choosing a desired point on the $\epsilon^2(\eta^2)$ curve, as illustrated in Fig. 1. For example, choosing $K_6^2 = 0.5$ results in this example in $\delta_6 = 0.034$ and $\epsilon = 3.715 \times 10^{-2}$.

The frequency responses of the optimal individual filters for the two extremal cases are compared in Fig. 4.

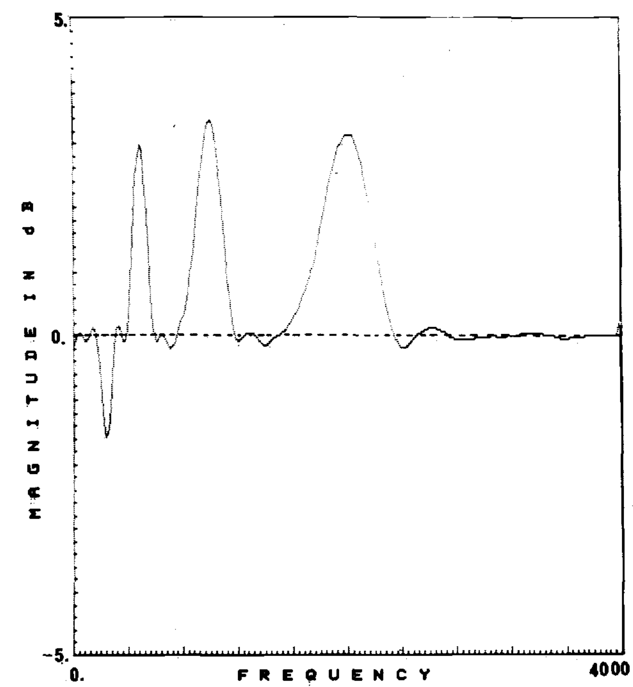
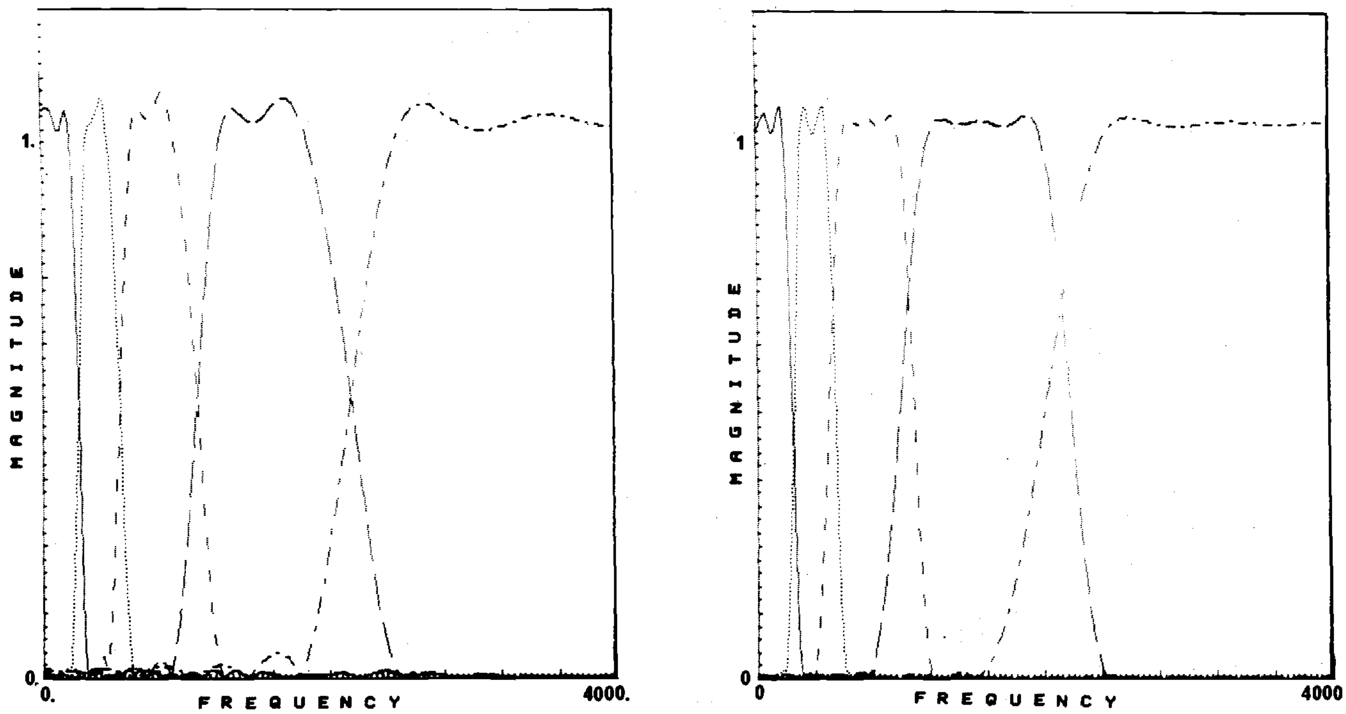


Fig. 3. Composite response error for the unconstrained design (solid line) and the constrained design (dashed line).

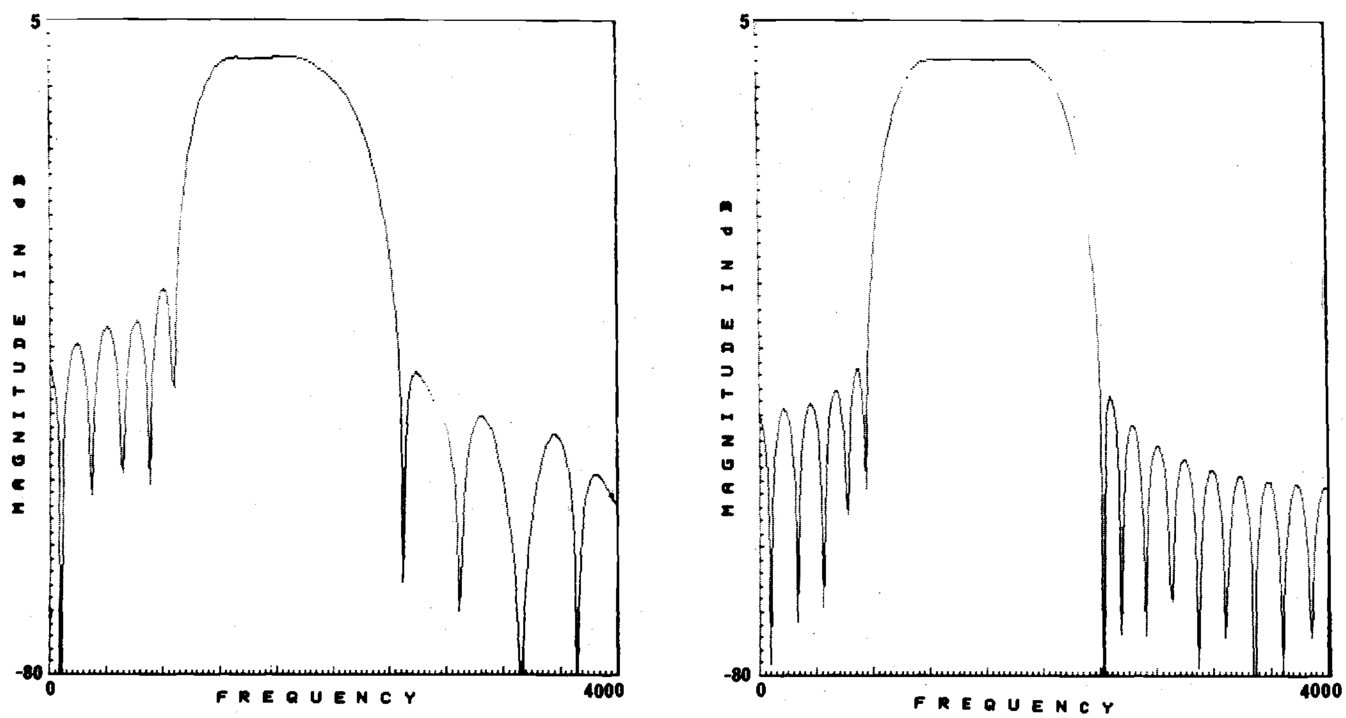
The frequency responses of the filters obtained in the flat composite response design are illustrated in Fig. 4(a), and the frequency responses of the filters obtained in the unconstrained design are in Fig. 4(b). Both are shown on a linear magnitude scale. For further comparison, the frequency responses of the fourth individual filter in these two designs are illustrated in Fig. 5(a) and (b), respectively, on a logarithmic magnitude scale. The values of δ_i obtained in the two extremal designs and the overall rmse ϵ are summarized in Table II.

It is significant that the two extreme values of ϵ are quite close to each other (last row in the table), whereas the values of δ_6 differ dramatically. Thus, with a moderate increase of the MSE, a flat composite response is obtained, instead of the poor composite response which results in the design which ignores composite response specifications.



(a) (b)

Fig. 4. (a) Frequency response of each of the five optimal filters for the constrained design (linear magnitude scale). (b) Frequency response of each of the five optimal filters for the uncontrained design (linear magnitude scale).



(a) (b)

Fig. 5. (a) Frequency response of the fourth optimal filter for the constrained design (in decibels). (b) Frequency response of the fourth optimal filter for the unconstrained design (in decibels).

VII. Conclusions

The above design example illustrates the strength of the new design method by obtaining the specified composite response, even when each filter in the bank has a different length. The composite response can be specified to be flat, as in the above example, or it can be any other desired response (e.g., when the sampling frequency of the input process is higher than the Nyquist rate, a low-pass type of composite response can be specified). The composite response can be specified as a constraint on the design, as in the above example, or in terms of an allowed tolerance. For the latter type of specification, some properties of the tradeoff curve that relates the overall performance to the allowed tolerance were illustrated. Among these properties are the monotonicity of this curve and its convexity, as well as a simple geometrical interpretation of the weight constant K_{N+1}^2 as the negative slope of the curve. Using an eigenvector decomposition scheme, one is able to compute this design curve and thus solve the design problem for any value of K_{N+1}^2, with complexity similar to that of the design with a specific tolerance value. Once the tradeoff curve is plotted, it helps in choosing the appropriate value of K_{N+1}^2.

The new method is very flexible in the sense that the individual filters in the bank need not be conventional FIR filters. They can be linear combinations of some predesigned realizable filters, and the new method optimizes the performance with respect to the coefficients of these combinations. Thus, generalized structures of FIR filters as suggested in [3] and [4] are applicable as well as conventional FIR. IIR structures are allowed, provided that the poles' locations are given and the optimization is on the zeros' locations represented by the coefficients a_{ik} in (1). In general, the optimal individual filters have complex coefficients and an arbitrary phase response. However, real coefficients and zero phase error can be achieved by fulfilling the conditions stated in Section III. These conditions are given for the general filter structures, with simplified versions corresponding to real filter banks composed of conventional FIR filters with linear phase. For example, the more general conditions can be applied when IIR components with approximately linear phase response are used, and the design goal is to get an improved magnitude response, without degrading the phase response, by means of linear combination of these components.

Special emphasis is given to the complexity of the design. This issue is very important since typically the number of coefficients in the filter bank is on the order of several hundred up to even several thousand. To illustrate that the new method is easily implemented on fairly small computers, we remark that the above design example, which involves an overall number of 415 coefficients, runs on a 16 bit machine, written in Fortran, in less than 1 min of CPU time.

In speech processing applications, the deterministic design approach is particularly suitable since there is no valid statistical characterization of the input processes. On the other hand, in many communication applications there are established statistical characterizations of the input signal and noise processes, so that the statistical approach is more suitable for setting the design specifications. Examples are the detection of frequency-hopping signals by means of filter banks and the design of filter banks for TDM/FDM systems.

Finally, we note that although the problem of designing a single FIR filter subject to linear constraints on its impulse response is not in the scope of this work, its solution can be derived as a special case in the mathematical framework presented here.

Appendix

In this appendix, we investigate the complexity of evaluating the elements of the matrix $\mathbf{V}$ that appear in lemma 1. In [17] the following method is applied for evaluating $\mathbf{V}$ as well as the values of $\{d_n\}_{n=1}^{M_{N+1}}$.

1) Compute the matrix $\mathbf{C} = \mathbf{R}_{N+1}\mathbf{T}$.

2) Solve the eigenvalue/eigenvector problem $\mathbf{C}\boldsymbol{u} = \lambda \boldsymbol{u}$.

It can be shown that for $\mathbf{R}_{N+1}$ and $\mathbf{T}$, which are both Hermitian, and $\mathbf{R}_{N+1}$ being p.d., the matrix $\mathbf{C}$ can be diagonalized.

Now, since $\mathbf{C}$ is a diagonalizable matrix, there exists a nonsingular matrix $\mathbf{U}$ (whose columns are the eigenvectors) such that $\mathbf{CU} = \mathbf{U}$ diag $\{d_1 \cdots d_{M_{N+1}}\}$. $\{d_n\}_{n=1}^{M_{N+1}}$ are therefore the eigenvalues of $\mathbf{C}$. It can be shown that for $d_n \neq d_m$, $\boldsymbol{u}_n^H \mathbf{R}_{N+1}^{-1} \boldsymbol{u}_m = \boldsymbol{u}_n^H \mathbf{T} \boldsymbol{u}_m = 0$.

3) If all $\{d_n\}_{n=1}^{M_{N+1}}$ values are distinct, then $\mathbf{V}$ is obtained from $\mathbf{U}$ by scaling the columns of $\mathbf{U}$ as follows: $\boldsymbol{v}_n = \boldsymbol{u}_n / (\boldsymbol{u}_n^H \mathbf{R}_{N+1}^{-1} \boldsymbol{u}_n)^{1/2}$.

4) If the eigenvalue d_n has $m \geq 1$ eigenvectors associated with it, a Grahm–Schmidt orthonormalization process on the subspace of dimension m of these eigenvectors will give the m columns of the matrix $\mathbf{V}$ corresponding to this eigenvalue. The orthonormalization process is with respect to the following norm of $\mathbb{C}^{M_{N+1}}$ defined by $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^H \mathbf{R}_{N+1}^{-1} \boldsymbol{y}$. For $\mathbf{R}_{N+1}$, which is a p.d. Hermitian matrix, this is a well-defined norm, and the case of $m = 1$ discussed in step 3) above is only a special case.

The complexity of the matrix multiplication in step 1) above is $\mathbf{O}(M_{N+1}^3)$. The complexity of the eigenvector/eigenvalue problem that is solved in step 2) is about $\mathbf{O}(M_{N+1}^3)$ using efficient numerical methods [18]. The complexity of the normalization process in step 3) or 4) is also $\mathbf{O}(M_{N+1}^3)$. Thus, the overall complexity of evaluating the elements of the matrix $\mathbf{V}$ is $\mathbf{O}(M_{N+1}^3)$. However, this task is certainly more complex than simply inverting an $M_{N+1} \times M_{N+1}$ matrix.

Note that for the special case of a desired flat composite response, $\mathbf{R}_{N+1} = I$, and step 1) is totally omitted. Futhermore, in step 2) $\mathbf{C} = \mathbf{T}$ is an Hermitian p.s.d matrix; thus, it has a unitary diagonalization $\mathbf{V}$, which is the matrix that appears in lemma 1 [steps 3) and 4) are omitted].

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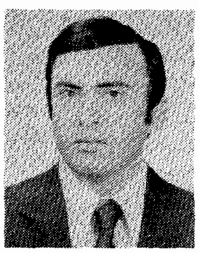

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