The Design of Optimal Uniform Filter Banks with Specified Composite Response

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***Abstract*—In this paper we prove that for suitable design specifications and a wide class of optimization criteria, the optimal complex filter bank with specifications on its composite response is composed of frequency translated versions of a prototype filter. In particular, this holds for the min–max and WMMSE (weighted minimum mean square error) criteria. As a result, a simplified design problem whose solution is an optimal prototype filter is formulated. This prototype is essentially an optimal FIR low-pass filter subject to linear constraints on its impulse response.**

For the WMMSE criterion, this characterization of the optimal filter bank results in a simplified version of the design method presented in [1]. For the min–max criterion, this characterization implies that there exists an optimal window, by which the window design method results in the optimal low-pass prototype. The optimal window design problem is formulated as a linear programming problem, and an approximate solution is derived using the Remez exchange algorithm.

For real filter banks in which each filter is composed of a pair of complex filters, the optimal filter bank is no longer composed of frequency translated versions of prototype filter. However, for efficient implementation, the prototype translation property may be part of the design specifications. For this reason, the optimal WMMSE prototype for a class of real filter banks is derived as well.

I. Introduction

In many applications, digital filter banks with specified composite response (usually flat, or having bandpass characteristics) are required. For example, filter banks are used in analyzing speech signals for speech recognition applications [2]. The flat composite response guarantees that the sum of the outputs of all the filters in the bank restores the original input signal so that no signal component is misrepresented.

The conventional filter banks in these applications are composed of filters that are FIR (finite impulse response) digital filters with linear phase, and either real or complex coefficients. Either the Remez exchange method for the design of optimal min–max FIR filters [4], or the statistical approach for the design of optimal Wiener FIR filters [5], can be used to design each filter independently of the other filters in the filter bank. However, direct application of these methods typically results in a poor composite response [1], [6].

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The design of a filter bank with specified composite response and minimal weighted mean square error was presented in [1]. The design of a filter bank, with specified composite response and minimal weighted L_∞ error, involves linear programming techniques similar to the one presented in [7]. In both cases, the number of variables, which is equal to the overall number of filter coefficients in the filter bank, can be well over 1000. This rules out the feasibility of the optimal L_∞ design method for many applications, and the optimal WMMSE design in [1] may also become quite complex.

The window method for the design of filter banks with specified composite response [8] is very popular due to its simplicity. However, longer filters are needed to meet the frequency response specifications, as compared to the min–max design [9], thus, this method is not optimal.

In the above discussion we assumed that the different filters in the filter bank are independent FIR filters. Therefore, implementation of a filter bank consisting of N FIR filters, each of them of length M, demands NM multiplications per input sample. However, if these FIR filters are frequency translated versions of a prototype filter, then an efficient implementation using the FFT algorithm requires only on the order of $N \log N + M$ multiplications per input sample [3]. This is the main reason that many of the digital filter banks used in practice are composed of frequency translated versions of a prototype filter, and they will be denoted in the sequel as prototype translated filter banks (PTFB).

In the next section we define the class of complex uniform design problems (CUDP) for which we prove that the optimal filter bank with specific composite response is a PTFB. As a result, an *equivalent*, simpler design problem whose solution is the optimal prototype filter is formulated. The optimal prototype filter is the optimal FIR low-pass filter (LPF) subject to linear constraints on its impulse response.

Section III presents the *simplified version* of the optimal WMMSE design method of [1] for the CUDP (i.e., the design of the optimal prototype LPF), and the issue of computational complexity is elaborated, whereas Section IV presents the design of the optimal *min–max* prototype for the CUDP. These two design algorithms complement each other, since each one is oriented toward different applications (for example, compare [4] and [5]). The optimal min–max prototype design is by means of linear programming techniques. In the case of a flat composite response specification, it is shown that the linear programming design method, presented in [10] for different applications, is also suitable for the design of this prototype filter.

However, the high complexity of the linear programming techniques limits the feasibility of this method to the design of filters with less than 100 coefficients [7]. Therefore, in Section V, it is shown that the design of the optimal min–max prototype for the CUDP when a flat composite response is specified can be done by the window method. This in itself does not lead to a significant reduction in the design complexity, since the optimal window has to be designed using linear programming techniques. However, based on the performance analysis of the window method for FIR design in [11], we present a much simpler method for designing a window which approximates the desired optimal window.

This approximation uses the Remez exchange algorithm, and the design problem is restated so that the solution is obtained via an available design program [4]. The approximate optimal window (AOW) design leads to superior (in the min–max sense) prototype filters than those obtained by other suboptimal methods [8], [12] as is demonstrated via a design example. For filters of short length, the AOW design is compared to the optimal solution, and results in a small degradation in performance. However, the complexity of the AOW design method is relatively very low compared to the optimal min–max design. Thus, the AOW is applicable to the design of prototype filters of length of even several hundred taps.

In many applications real outputs are desired, and it cannot be obtained by a PTFB. Thus, usually proper pairs of filters in the PTFB are added together yielding real outputs, with an efficient implementation. However, unlike the case of CUDP, in general, this type of filter banks is not optimal, and thus it degrades the performance as compared to the *optimal real filter bank* having the same specifications. Nevertheless, in order to take advantage of the efficient implementation, this realization is frequently imposed as part of the design specifications. Therefore, we derive in Section VI the optimal WMMSE prototype for this case.

Conclusions are drawn in the last section.

II. The Optimality of Prototype Translated Filter Banks for Complex Uniform Design Problems

We begin with introducing the complex uniform design problems (CUDP) in a general framework, which enables a unified treatment for both the WMMSE and the min–max criteria. We then prove that for any CUDP, at least one optimal solution is a PTFB, and as a consequence formulate an equivalent, simpler design problem.

A. Complex Uniform Design Problems

A) The filter bank is composed of N individual digital filters. The ith filter is an FIR filter with M complex coefficients (denoted by $\{a_{ik}\}_{k=-L}^{L}$, where $L \triangleq (M-1)/2$), and its frequency response is

$$H_i(f) \triangleq \sum_{k=-L}^{L} a_{ik} e^{-j2\pi kf}. \tag{1}$$

Remark: For ease of presentation we use a noncasual version of the filter bank, casuality can be obtained by a proper delay.

B) The *desired* frequency response of the ith filter is denoted by $D_i(f)$, $i = 1, \cdots, N$. The frequency weighted error between this desired frequency response and the frequency response of the corresponding filter is

$$E_i(f) \triangleq W_i(f)\left(D_i(f) - H_i(f)\right) \tag{2}$$

where $W_i(f) \geq 0$ is a weight function which is nonzero on a set of positive measure. Thus, the ith filter response error is defined as a suitable norm (denoted by $\| \ \|$) of the periodic function $E_i(f)$, i.e.,

$$\delta_i \triangleq \left\| E_i(f) \right\|. \tag{3}$$

For example, in the WMMSE design the L_2 norm is used, whereas in the min–max design the L_∞ norm is used.

C) The composite response of the filter bank [denoted by $H_c(f)$] is the sum of the responses of all N filters, i.e., $H_c(f) \triangleq \Sigma_{i=1}^{N} H_i(f)$. The composite response error is defined as

$$\delta_c \triangleq \left\| E_c(f) \right\| \tag{4a}$$

where

$$E_c(f) \triangleq W_c(f)\left(D_c(f) - H_c(f)\right) \tag{4b}$$

with $D_c(f)$ being the desired composite response, and $W_c(f) \geq 0$ the composite response weight function which is nonzero on a set of positive measure. For example, a flat composite response is specified by $D_c(f) = 1$.

D) The overall performance of the filter bank is measured by a seminorm (denoted by $\| \ \|_T$), which takes into account the response errors of all the filters. Let $\boldsymbol{\delta} \in \mathbb{R}_+^N$ be the vector, whose ith component is δ_i—the error of the ith filter, then the overall error is defined as

$$\epsilon \triangleq \left\| \boldsymbol{\delta} \right\|_T \tag{5}$$

where $\| \ \|_T$ is defined on $\mathbb{R}_+^N$, the space of N-dimensional real vectors with nonnegative components, and an increase in the errors of some of the individual filters never decreases the overall error, i.e., for every $\boldsymbol{\delta}$, $\boldsymbol{\Delta\delta} \in \mathbb{R}_+^N$, $\| \boldsymbol{\delta} + \boldsymbol{\Delta\delta} \|_T \geq \| \boldsymbol{\delta} \|_T$.

E) The design of an optimal filter bank with a specified composite response is the solution of the following constrained optimization problem:

$$\underset{\{a_{ik}\}_{i=1,k=-L}^{N\ \ L} | \delta_c \leq \eta}{\text{Min}} \{\epsilon\} \tag{6}$$

where η is the allowed tolerance of the composite response error. The existence of a solution (not necessarily a unique one) to this design problem for large enough η was shown elsewhere [16].

Definition: A complex uniform design problem (CUDP) is characterized by

$$D_i(f) = D_o\left(f + \Delta + \frac{i}{N}\right); \quad i = 1, \cdots, N \quad (7a)$$

$$D_c(f) = D_c\left(f + \frac{i}{N}\right); \quad i = 1, \cdots, N \quad (7b)$$

$$W_i(f) = W_o\left(f + \Delta + \frac{i}{N}\right); \quad i = 1, \cdots, N \quad (7c)$$

$$W_c(f) = W_c\left(f + \frac{i}{N}\right); \quad i = 1, \cdots, N \quad (7d)$$

$$\|\boldsymbol{\delta}^{(l)}\|_T = \|\boldsymbol{\delta}\|_T; \quad \text{for any } \boldsymbol{\delta} \in \mathbb{R}_+^N \quad (7e)$$

$$\|A(f)\| = \left\|A\left(f + \frac{i}{N}\right)\right\| = \left\|A\left(f + \frac{i}{N} + \Delta\right)\right\|;$$

$$\text{for any function } A(f) \text{ and } i = 1, \cdots, N \quad (7f)$$

where $\boldsymbol{\delta}^{(l)}$, $l = 1, \cdots, N$, denotes the vector obtained by a cyclic shift of the components of the vector $\boldsymbol{\delta}$, i.e., $\delta_i^{(l)} = \delta_{(i+l)\text{mod}N}$. The interpretation of (7a)–(7f) is that the individual filters specifications are translated versions of a prototype specification, the composite response specifications are invariant under frequency translation, and each one of the filters has the same contribution to the error measure ϵ.

B. The Equivalent Design Problem

Definition: A prototype translated filter bank (PTFB) is a filter bank with the property: $H_i(f) = H_o(f + i/N + \Delta)$, $i = 1, \cdots, N$, for some value of Δ and some frequency response of an FIR filter of length M, which is denoted by $H_o(f)$.

Theorem 1: Any CUDP has as least one optimal solution which is a PTFB.

The proof is given in the Appendix.

Theorem 2: For CUDP of odd length (L is an integer value), the optimal PTFB is obtained as follows.

a) Solve the following *prototype* FIR filter design problem:

$$\underset{\{a_{ok}\}_{k=-L}^{L}}{\text{Min}} \left\| W_o(f)\left(D_o(f) - \sum_{k=-L}^{L} a_{ok}e^{-j2\pi fk}\right)\right\| \quad (8a)$$

subject to the following constraint imposed by the composite response specification:

$$\delta_c \triangleq \left\| W_c(f - \Delta)\left(D_c(f - \Delta) - N \sum_{m=\lceil -L/N \rceil}^{\lfloor L/N \rfloor} a_{o(mN)}e^{-j2\pi fmN}\right)\right\| \leq \eta. \quad (8b)$$

This is the *equivalent design problem* with which we shall deal in the succeeding sections.

b) The coefficients of the PTFB filters are obtained from the optimal prototype filter [the solution of (8)] by

$$a_{ik} \triangleq a_{ok}e^{-j2\pi(i/N+\Delta)k} \quad i = 1, \cdots, N; \quad k = -L, \cdots, L. \quad (9)$$

The proof is given to the Appendix.

Remarks:

1) Since Δ in the PTFB and CUDP definitions is arbitrary, its value can always be chosen so that the prototype filter is a low-pass filter.

2) For even length CUDP a similar equivalent problem can be derived, where only (8b) is properly modified. We omit the details here for simplicity of the presentation.

3) Since $\| \ \|_T$ *does not affect* the equivalent problem, we can ignore it in the sequel!

4) For $D_o(f)$, $W_o(f)$, $D_c(f - \Delta)$, and $W_c(f - \Delta)$ which are real even functions, and $\|A(\cdot)\| = \|A^*(\cdot)\| = \|A^*(-\cdot)\|$, for every periodic function $A(\cdot)$, it can be shown (cf. [16]) that the optimal prototype filter has zero phase and real coefficients, i.e., $a_{ok} = a_{ok}^* = a_{o(-k)}$. Therefore, the delayed-by-L-samples casual version of the optimal PTFB will be composed of FIR filters with common linear phase. The proof is beyond the scope of this paper.

III. The Optimal WMMSE Solution for CUDP

The WMMSE design of general filter banks with specified composite response has been presented in [1]. Here we concentrate on the WMMSE CUDP, and as a consequence of theorem 2, obtain a simplified algorithm for the design of the optimal (in the WMMSE sense) solution, which has the structure of a PTFB.

The WMMSE CUDP is a filter bank design problem, for which (7a)–(7d) holds, and the L_2 norm is used, i.e.,

$$\|A(f)\| \triangleq \left[\int_{-0.5}^{0.5} |A(f)|^2 df\right]^{1/2}; \quad \text{for any periodic function } A(f). \quad (10)$$

It is clear that this norm satisfies (7f). Therefore, applying theorem 2, the equivalent design problem for odd length WMMSE CUDP is

$$\underset{\{a_{ok}\}_{k=-L}^{L}}{\text{Min}} \ \epsilon^2 \triangleq N \int_{-0.5}^{0.5} |W_o(f)|^2 \left| D_o(f) - \sum_{k=-L}^{L} a_{ok}e^{-j2\pi fk}\right|^2 df \quad (11a)$$

subject to

$$\delta_c^2 \triangleq \int_{-0.5}^{0.5} |W_c(f - \Delta)|^2 \left| D_c(f - \Delta) - N \sum_{m=\lceil -L/N \rceil}^{\lfloor L/N \rfloor} a_{o(mN)}e^{-j2\pi fmN}\right|^2 df \leq \eta^2. \quad (11b)$$

This convex programming problem is equivalent to [13, sec. 4.5]:

$$\underset{\{a_{ok}\}_{k=-L}^{L}}{\text{Min}} \quad \{\epsilon^2 + K^2 \delta_c^2\}. \tag{12}$$

For some value of $K(\eta)$. Differentiating (12) with respect to the unknown variables and rearranging the resulting equations leads to the solution:

$$\boldsymbol{a}_o = R_o^{-1}[\boldsymbol{d}_o + H^T \boldsymbol{q}] \tag{13a}$$

where $\boldsymbol{a}_o \in \mathbb{C}^M$ is the vector whose elements are the coefficients of the prototype filter. The elements of the positive definite Hermitian $M \times M$ matrix R_o are

$$R_o(m, k) = \int_{-0.5}^{0.5} |W_o(f)|^2 e^{j2\pi f(m-k)} df; \quad m, k = -L, \cdots, L, \tag{13b}$$

and the elements of $\boldsymbol{d}_o \in \mathbb{C}^M$ are

$$d_o(m) = \int_{-0.5}^{0.5} |W_o(f)|^2 D_o(f) e^{j2\pi fm} df; \quad m, k = -L, \cdots, L. \tag{13c}$$

The elements of the $Q \times M$ matrix H (with $Q = 2\lfloor L/N \rfloor + 1$) are

$$H(m, k) = \begin{cases} 1 & k = mN \\ 0 & \text{elsewhere;} \end{cases} \quad m = -L_Q, \cdots, L_Q, \quad k = -L, \cdots, L \tag{13d}$$

where

$$L_Q \triangleq \left(\frac{Q-1}{2}\right) = \left\lceil \frac{L}{N} \right\rceil.$$

The vector $\boldsymbol{q} \in \mathbb{C}^Q$ is a correction vector due to the composite response specification and is defined as

$$\boldsymbol{q} = \left[\frac{1}{K^2} R_c^{-1} + NHR_o^{-1}H^T\right]^{-1} (R_c^{-1}\boldsymbol{d}_c - NHR_o^{-1}\boldsymbol{d}_o). \tag{13e}$$

where the elements of the positive definite Hermitian $Q \times Q$ matrix R_c are

$$R_c(m, k) = \int_{-0.5}^{0.5} |W_c(f - \Delta)|^2 e^{j2\pi f(m-k)} df; \quad m, k = -L_Q, \cdots, L_Q, \tag{13f}$$

and the elements of $\boldsymbol{d}_c \in \mathbb{C}^Q$ are

$$d_c(m) = \int_{-0.5}^{0.5} |W_c(f - \Delta)|^2 D_c(f - \Delta) e^{j2\pi fmN} df; \quad m = -L_Q, \cdots, L_Q. \tag{13g}$$

Equations (13a)–(13g) describe the optimal prototype given $K(\eta)$, and are the simplified version of (10)–(15) in [1]. The overall error (ϵ^2) and the composite response error (δ_c^2) of the optimal PTFB are

$$\epsilon^2 = N\left[\int_{-0.5}^{0.5} |W_o(f) D_o(f)|^2 df - \boldsymbol{d}_o^* R_o^{-1} \boldsymbol{d}_o + \boldsymbol{q}^* H^T R_o^{-1} H \boldsymbol{q}\right] \tag{14a}$$

$$\delta_c^2 = \int_{-0.5}^{0.5} |W_c(f) D_c(f)|^2 df - \boldsymbol{d}_c^* R_c^{-1} \boldsymbol{d}_c + \boldsymbol{q}^* R_c^{-1} \boldsymbol{q} \tag{14b}$$

where $\boldsymbol{v}^*$ denotes conjugate transposition of $\boldsymbol{v}$. The value of $K(\eta)$ is obtained, similar to [1, equations (16)–(24)], by simultaneously diagonalization of R_c^{-1} and $T \triangleq NHR_o^{-1}H^T$.

Usually $|W_o(f)|^2$, $|W_c(f - \Delta)|^2$, $D_o(f)$, and $D_c(f - \Delta)$ are piecewise constant functions, and all the integrals appearing in (13) and (14) can be evaluated analytically. Thus, the complexity of the design is $O(M^2 + \alpha(M/N)^3)$. $O(M^2)$ operations are needed for the inversion of the M-dimensional Toeplitz matrix R_o, and for obtaining the solution $\boldsymbol{a}_o$ via (13a). $O(\alpha(M/N)^3)$ operations are needed for the simultaneous diagonalization of the $Q \times Q$ symmetric matrices R_c^{-1} and $NHR_o^{-1}H^T$, where $Q \cong M/N$, and $\alpha > 1$ represents the relative complexity of unitary diagonalization of a matrix compared to its inversion. In comparison, the original algorithm of [1] involves inversion of N Toeplitz matrices, then a simultaneous diagonalization of two $M \times M$ matrices, and thus its complexity is $O(NM^2 + \alpha M^3)$. For example, the design of a typical digital filter bank with 32 filters, each of them an FIR filter of length 256, takes a few CPU seconds on a 16 bit computer using the new algorithm, whereas a direct design requires the solution of an optimization problem with 8192 variables!

IV. The Optimal Min–Max Solution for CUDP

Optimal min–max design of filter banks with specified composite response involves linear programming techniques. The linear program for each filter is of M unknown variables and P linear constraints, where the continuous frequency response error of this filter is uniformly sampled with a sampling interval of $1/P$ [7], [14]. For the whole filter bank design, the overall linear programming effort involves $MN + 1$ unknown variables and $P(N + 1)$ linear constraints including the composite response specifications. Each iteration of the linear programming algorithm involves $(N + 1)$ FFT's of dimension M each, followed by interpolations by a factor of P/M [7]. This step which has a complexity of $O((N + 1) M \log_2 M + (N + 1)\alpha P)$ operations per iteration determines the complexity of the algorithm, where α is the number of operations per output sample of the interpolation filter. For typical values of $N = 32$, $M = 256$, and $P = 1024$ this algorithm is quite complex.

The min-max CUDP is a filter bank design problem, for which (7a)–(7d) hold, and the L_∞ norm is used. Since this norm satisfies (7f), it follows from theorem 2 that for odd length min-max CUDP, the solution is a PTFB whose prototype filter is obtained by

$$\operatorname*{Min}_{\{a_{ok}\}_{k=-L}^{L}} \operatorname*{Sup}_{f\in[-0.5,0.5]} \left\{ |W_o(f)| \left| D_o(f) - \sum_{k=-L}^{L} a_{ok} e^{-j2\pi fk} \right| \right\} \tag{15a}$$

subject to

$$\operatorname*{Sup}_{f\in[-0.5,0.5]} \left\{ |W_c(f-\Delta)| \left| D_c(f-\Delta) - N \sum_{m=\lceil -L/N \rceil}^{\lfloor L/N \rfloor} a_{o(mN)} e^{-j2\pi fmN} \right| \right\} \le \eta. \tag{15b}$$

This *equivalent design problem* involves only M variables and $2P$ constraints, thus, the complexity of the design is reduced by at least a factor of N. The optimal prototype filter has real coefficients and linear phase provided that the conditions presented in Section II are fulfilled, and then additional reduction in the design complexity is possible.

There exists a solution of (15a)–(15b), which is obtained with equality in (15b) provided that $\eta \ge \eta_m$, and the minimal composite response error η_m is the solution of

$$\eta_m = \operatorname*{Min}_{\{b_m\}_{m=-L_Q}^{L_Q}} \left\{ \operatorname*{Sup}_{f\in[-0.5,0.5]} |W_c(f-\Delta)| \left| D_c(f-\Delta) - N \sum_{m=-L_Q}^{L_Q} b_m e^{-j2\pi fNm} \right| \right\} \tag{16}$$

with $L_Q \triangleq \lfloor L/N \rfloor$. The proof which is beyond the scope of this paper can be found in [16].

The design of the optimal min-max prototype for $\eta = \eta_m$ (i.e., minimal composite response error) can thus be done in the following two steps:

a) solve (16) to obtain η_m and $\{b_m\}_{m=-L_Q}^{L_Q}$,

b) substitute $a_{o(mN)} = b_m$, and solve (15a) to complete the impulse response of the prototype filter.

Each one of these steps involves a linear programming solution of a constrained FIR digital filter design problem, similar to the one presented in [10]. For the particular case of a flat composite response (i.e., $D_c(f) = 1$), the unique solution of (16) is $b_m = 1/N\,\delta(m)$, with $\eta_m = 0$. Thus, the optimal prototype for a *specified flat composite response* with zero tolerance is essentially a Nyquist filter that is designed as suggested in [10]. The linear programming approach is quite satisfactory for short length prototype designs (typically, for $M \le 100$). However, for values of M, which are several hundreds, this approach becomes impractical.

Note: For flat composite response, the unique solution of (16) is *independent of* Δ. Thus, Δ, which is the center frequency of the Nth filter, does not affect the optimal prototype design, nor the performance of the resulted PTFB. Therefore, in the next section (which is devoted to the design for flat composite response specification), and the examples therein, we ignore this parameter.

V. Approximate Optimal Window Design

A. Derivation of the Approximate Optimal Window

The following theorem (which is proven in the Appendix) relates the solution for an important subclass of these min-max CUDP with the well-known window method for the design of FIR filter banks (cf. [8]).

Theorem 3: The solution of any odd length min-max CUDP with $D_o(f)$ which is an ideal LPF of bandwidth $1/2N$, a flat composite response specification ($D_c(f) = 1$), and zero tolerance ($\eta = \eta_m = 0$), can be obtained with the window method by a proper choice of the window sequence. Furthermore, for any window sequence with $w_o = 1$, this composite response specification is fulfilled.

We shall concentrate on this subclass of CUDP and efficiently design a good approximation of the optimal solution using this theorem. Since any window sequence with $w_o = 1$ results in a flat composite response, the optimal window [for which $\{a_{ok}\}_{k=-L}^{L}$ is a solution of (15a)] is any solution of:

$$\operatorname*{Min}_{\substack{\{w_k\}_{k=-L}^{L} \\ w_o=1}} \operatorname*{Sup}_{f\in[-0.5,0.5]} \left\{ |W_o(f)| \left| D_o(f) - \sum_{k=-L}^{L} d_o(k)\, w_k e^{-j2\pi fk} \right| \right\} \tag{17}$$

where $\{d_o(k)\}_{k=-L}^{L}$ are the coefficients of the desired (infinite) impulse response, i.e.,

$$d_o(k) \triangleq \int_{-0.5}^{0.5} D_o(f)\, e^{j2\pi fk} df = \begin{cases} \dfrac{1}{\pi k} \sin \dfrac{\pi k}{N} & k \neq 0 \\ \dfrac{1}{N} & k = 0. \end{cases} \tag{18}$$

For $|W_o(f)| = |W_o(-f)|$, there is always an optimal window with zero-phase and real coefficients (cf. [16]). Restricting the solution of the design problem to have these properties, the optimal window is the solution of:

$$\operatorname*{Min}_{\{w_k\}_{k=1}^{L}} \left\{ \delta_w \triangleq \operatorname*{Sup}_{f\in[0.,0.5]} |W_o(f)| \left| D_o(f) - \frac{1}{N} - \sum_{k=1}^{L} 2\frac{w_k}{\pi k} \sin\left(\frac{\pi k}{N}\right) \cos(2\pi fk) \right| \right\}. \tag{19}$$

This problem can be solved by linear programming as in [10]. It seemed at first that this is a Chebyshev approximation problem, and thus, the Remez exchange method can be applied. This is not true, since the Chebyshev approximation problem with unknown coefficients $a_{ok} \triangleq w_k d_o(k)$ results in general with $a_{o(mN)} \neq 0$, contradicting the constraints $d_o(mN) = 0$. For this reason, the method presented in [12] is suboptimal. Similar to [11], we represent the error induced in the window method as follows:

$$\delta_w = \text{Max}\left\{ \sup_{f\in[0,1/2N]} |W_o(f)| \left| J_w\left(\frac{1}{2N} - f\right) + J_w\left(\frac{1}{2N} + f\right)\right|, \sup_{f\in[1/2N,0.5]} |W_o(f)| \left| J_w\left(f - \frac{1}{2N}\right) + J_w\left(1 - \frac{1}{2N} - f\right)\right| \right\} \quad (20a)$$

where

$$J_w(f) \triangleq \int_f^{0.5} \left\{1 + \sum_{k=1}^{L} 2w_k \cos(2\pi\vartheta k)\right\} d\vartheta = (0.5 - f) - \sum_{k=1}^{L} \frac{w_k}{\pi k} \sin(2\pi f k). \quad (20b)$$

In [15], the following approximation of δ_w (denoted by $\hat{\delta}_w$) is used:

$$\hat{\delta}_w \triangleq \text{Max}\left\{ \sup_{f\in[0,1/2N]} |W_o(f)| \,\text{Max}\left[\left|J_w\left(\frac{1}{2N} - f\right)\right|, \left|J_w\left(\frac{1}{2N} + f\right)\right|\right], \sup_{f\in[1/2N,0.5]} |W_o(f)| \,\text{Max}\left[\left|J_w\left(f - \frac{1}{2N}\right)\right|, \left|J_w\left(1 - \frac{1}{2N} - f\right)\right|\right]\right\}. \quad (21)$$

It is easily verified that $\delta_w \leq 2\hat{\delta}_w$ for any window sequence. Furthermore, in [11] and [15], this approximation was used for most of the known window sequences and for many design examples, always leading $\delta_w \geq \hat{\delta}_w$. Let the approximate optimal window (AOW) denote the sequence with minimal value of $\hat{\delta}_W$. We therefore expect the error of the PTFB designed using the AOW sequence to be within a factor of two from the minimal error. Combining (20b) and (21) and rearranging the resulting expression, the AOW is the solution of

$$\min_{\{w_k\}_{k=1}^{L}} \left\{ \sup_{\vartheta\in[0,0.5]} \left[|\hat{W}(\vartheta)| \left| 0.5 - \vartheta - \sum_{k=1}^{L} \frac{w_k}{\pi k} \sin(2\pi\vartheta k)\right|\right]\right\} \quad (22a)$$

where

$$|\hat{W}(\vartheta)| \triangleq \text{Max}\left\{\left|W_o\left(\frac{1}{2N} - \vartheta\right)\right|, \left|W_o\left(\frac{1}{2N} + \vartheta\right)\right|, \left|W_o\left(\vartheta - \frac{1}{2N}\right)\right|, \left|W_o\left(1 - \frac{1}{2N} - \vartheta\right)\right|\right\} \quad (22b)$$

In (22b) we interpret $|W_o(\lambda)|$ as zero for $\lambda < 0$ or $\lambda > 0.5$. Unlike the original problem stated in (19), *this is a Chebyshev approximation problem*, thus, the Remez exchange method can be applied, and the AOW is easily obtained even for filter lengths of several hundreds. Solving (22) with the Remez exchange method appears to require a special program. However, the available program in [4] is suitable for this purpose by applying the following algorithm.

a) Design an optimal (in the min-max sense) *differentiator* of length M for the weight function $|\hat{W}(0.5 - \vartheta)|$ using [4] with an absolute error criterion.

b) Let $\{a_L, \cdots, a_1, 0, -a_1, \cdots, -a_L\}$ be the coefficients of this optimal differentiator, then the desired window sequence is given by $w_k \triangleq 2\pi k(-1)^{k+1} a_k$, $k = 1, \cdots, L$.

Remarks:

1) The AOW design method was derived for odd length min-max CUDP. For the even length case, a similar derivation leads to the definition of the AOW [which is the argmin of $\hat{\delta}_w$ defined in (21)] as the solution of

$$\min_{\{w_k\}_{k=1}^{M/2}} \left\{ \sup_{\vartheta\in[0,0.5]} \left[|\hat{W}(\vartheta)| \left| 0.5 - \sum_{k=1}^{M/2} \frac{2w_k}{\pi(2k-1)} \sin \pi\vartheta(2k-1)\right|\right]\right\}, \quad (23)$$

with $|\hat{W}(\vartheta)|$ defined by (22b). The expression above results from the analog of (20b) for even length sequences. The available program in [4] can be used to solve (23) using the following algorithm.

a) Design an optimal (in the min-max sense) *Hilbert transformer* of length M for the weight function $|\hat{W}(\vartheta)|$ using the program in [4].

b) Let $\{a_{M/2}, \cdots, a_1, -a_1, \cdots, -a_{M/2}\}$ be the coefficients of this filter, then the desired window sequence is given by $w_k \triangleq \pi(k - \frac{1}{2}) a_k$, $k = 1, \cdots, M/2$.

2) For a general (nonuniform) filter bank design problem with specified flat composite response, the AOW can be defined in a similar manner, where the weight function $|\hat{W}(\vartheta)|$ given in (22b) is properly changed. The AOW results in an approximation to the optimal filter bank among those designed by the window method. Therefore, it usually leads to a superior performance as compared to conventional window sequences. However, for nonuniform filter banks, the AOW design is not necessarily an approximation to the *optimal (in the min-max sense) filter bank*. For this reason we concentrated on CUDP, although the AOW may be useful in the more general case as well.

B. Design Examples

We compare the design using the AOW sequence to the optimal min-max prototype via the design example 1 in [10]. This example corresponds to CUDP with 8 FIR filters, each one of them has 39 taps. The passband of the prototype is $f \in [0, 0.10625]$, whereas its stopband is $f \in$

TABLE I
TYPICAL DESIGN EXAMPLE

	KW [8]		AOW		TMX [12]		DMX [4]		
	A_p	A_s	A_p	A_s	A_p	A_s	A_p	A_s	A_c
$W = 1$	0.22	38.10	0.25	38.45	0.14	41.57	0.14	41.68	0.03
$W = 10$	0.22	38.10	1.10	46.68	1.03	30.25	0.54	50.08	4.61
$W = 50$	0.22	38.10	3.09	51.22	—	—	1.08	58.18	8.73

A_p—Passband ripple in decibels.
A_s—Stopband attenuation in decibels.
A_c—Composite response ripple in decibels.

[0.14375, 0.5]. The desired passband deviation equals the desired stopband deviation [i.e., $W_o(f)$ is as in Fig. 5(b) with $w = 1$]. The optimal min-max prototype has a passband ripple of 0.35 dB, and stopband attenuation of 33 dB. The AOW design results in a passband ripple of 0.43 dB and stopband attenuation of 32 dB. Thus, the degradation due to the suboptimally of the AOW design is very small. Similar results have been obtained for other design examples. For longer filters (typically above 100 taps), the optimal min-max design is too complex. Thus, we compare the design using the AOW sequence to various suboptimal methods, namely, the conventional window method in [8] using the Kaiser window (denoted by KW), the suboptimal min-max design of [12] (denoted by TMX), and the unspecified composite response design in [4] (denoted by DMX). The comparison is via a typical CUDP. Since TMX is applicable only to odd length filters, we consider a bank of $N = 16$ filters, having each an impulse response of length $M = 123$ samples. A flat composite response is specified, i.e., $D_c(f) = 1$, and the desired prototype frequency response is that of an ideal LPF of bandwidth $\frac{1}{32}$ (i.e., $D_o(f) = 1$ for $|f| \leq \frac{1}{32}$ and zero elsewhere). All four design methods are applied to design a PTFB. For these specifications KW, AOW, and TMX guarantee a flat composite response ($\delta_c = 0$). The prototype weight function is

$$|W_o(f)| \triangleq \begin{cases} 1 & |f| \leq F_p \\ 0 & F_p < |f| < F_s \quad F_p < \frac{1}{32} < F_s \\ W & F_s \leq |f| \leq 0.5. \end{cases} \tag{24}$$

The transition bandwidth is $F_s - F_p \triangleq \frac{0.55}{32}$. For KW, DMX, and TMX $(F_s + F_p)/2 = \frac{1}{32}$, whereas for the AOW $(F_s + F_p)/2$ is optimally set according to [11]. Table I summarizes the performance obtained for three typical values of $W = 1, 10, 50$. As expected, the DMX design has the smallest passband ripple, and largest stopband attenuation. However, it results in a poor composite response, with a ripple of up to 8.73 dB for $W = 50$. For $W = 1$ all four methods results in a similar performance, whereas for $W >> 1$ the AOW is certainly preferred on TMX and KW. Since TMX is applicable only for $(N - 1) \geq W \geq 1$ [12], it is not used for $W = 50$. Figs. 1–4 illustrate the shape of the response of the prototype filter which results in the DMX, TMX, KW, and AOW designs, respectively, for $W = 10$. In Fig. 1, the composite response of the DMX filter bank is also illustrated.

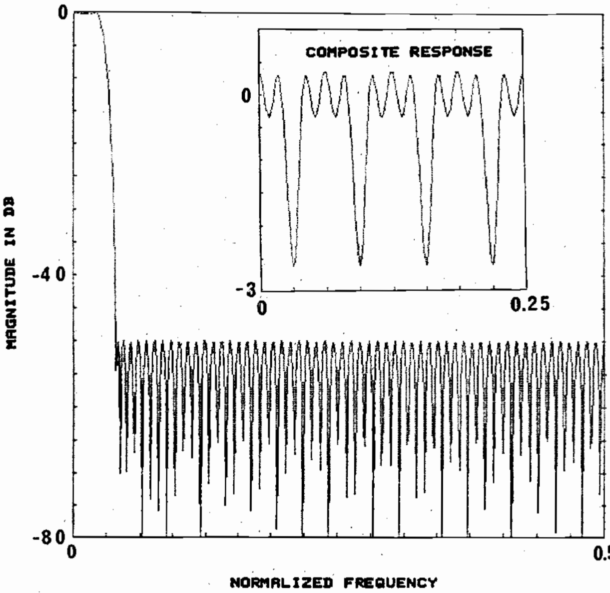

Fig. 1. Frequency response of the low-pass prototype filter, and the resulting composite frequency response when using the DMX method.

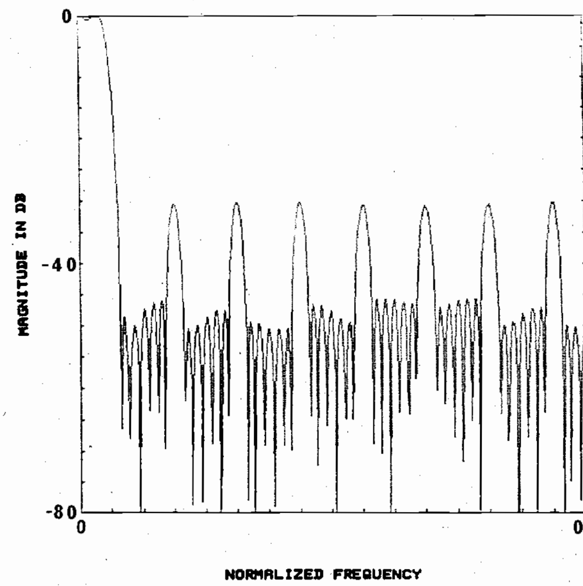

Fig. 2. Frequency response of the low-pass prototype filter using the TMX method.

It is obvious in Fig. 2 that the low stopband attenuation of the TMX results due to the spurious peaks in the frequency bands around $\frac{1}{32} + m/16$. $m = 1, \cdots, 14$. If these bands are excluded from the stopband by proper change of $|W_o(f)|$, similar performance to the AOW is achieved. Figs. 3 and 4 illustrate the shape of the filters designed using KW and AOW, respectively. Note that the AOW results in a nearly equiripple response.

VI. Real WMMSE Uniform Filter Banks

In many applications, real outputs are desired while preserving the efficient implementation of the PTFB. This is done by adding together the proper pair of outputs of

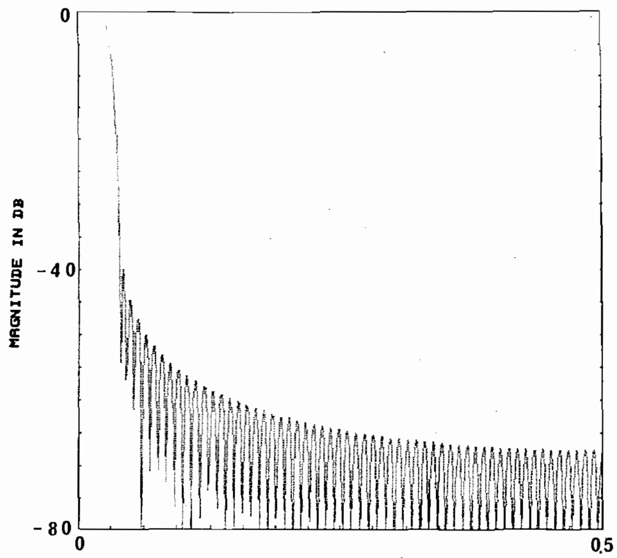

Fig. 3. Frequency response of the low-pass prototype filter using the KW method.

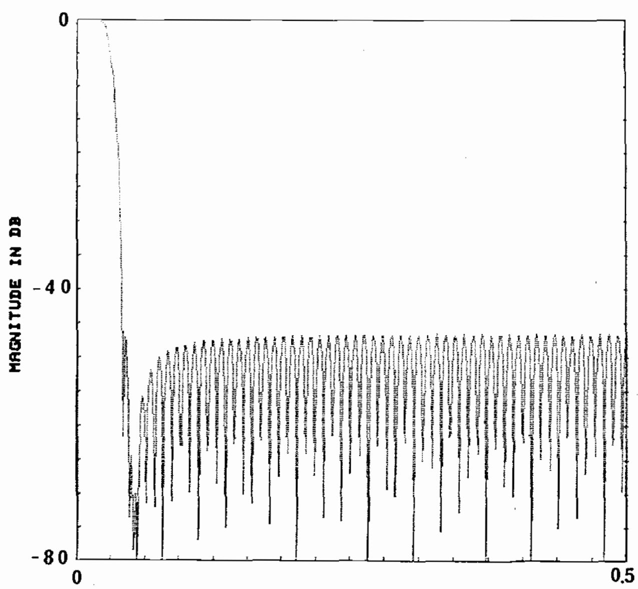

Fig. 4. Frequency response of the low-pass prototype filter using the AOW method.

the PTFB. The following result is an immediate consequence of (9).

Lemma 1: Adding pairs of outputs of a PTFB with a real prototype filter having a zero-phase response results in a *real* filter bank if and only if $\Delta \equiv 0 \bmod (1/2N)$.

Two different structures of real filter banks can thus be obtained from the same real prototype filter. The first one [denoted as structure (A)] corresponds to $\Delta = 0$ with the ith and $(N - i)$th outputs being added, yielding $\lceil N + 1/2 \rceil$ real filters; whereas in the second structure [denoted by (B)], $\Delta = 1/2N$ and the ith and $(N - 1 - i)$th outputs are added, yielding $\lceil N/2 \rceil$ real filters. The main difference between these two structures is in the bandwidth of the first (low-pass) filter in the resulting real filter bank.

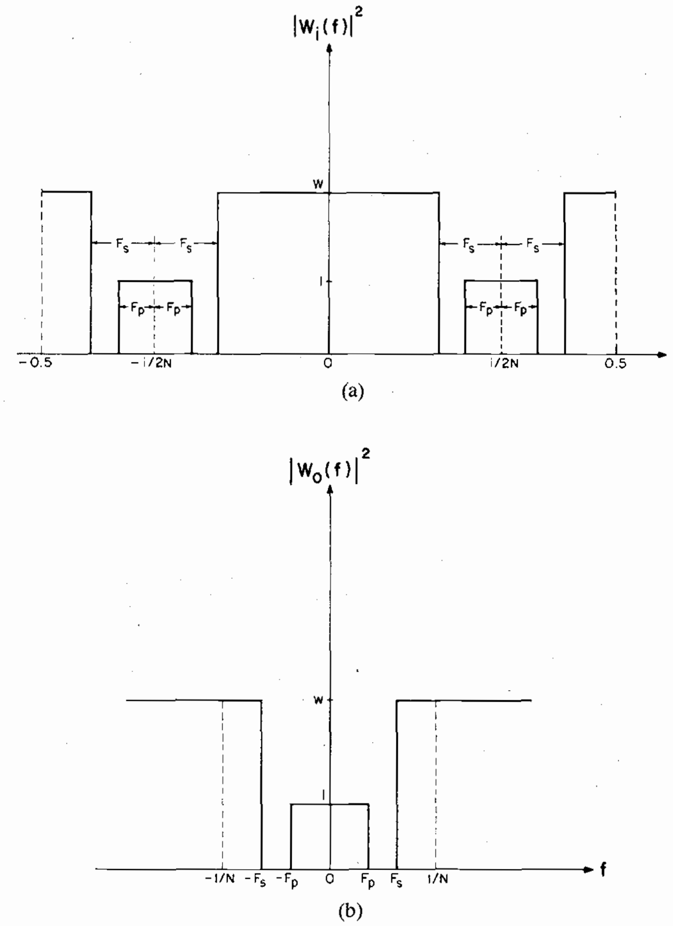

Fig. 5. (a) The weight function of the ith filter in an RUDP. (b) The weight function of the CUDP corresponding to Fig. 5(a).

Both structures are not suitable for a CUDP but rather for a real uniform design problem (RUDP) in which (7b), (7d), (7e), and (7f) hold, whereas (7a) is replaced by

$$D_i(f) = C_i\left[D_o\left(f + \frac{i}{2N}\right) + D_o\left(f - \frac{i}{2N}\right)\right], \quad \text{(25a)}$$

and (7c) is replaced by

$$|W_i(f)|^2 = \begin{cases} 1 & \left|f - \frac{i}{2N}\right| \le F_p \\ & \text{or } \left|f + \frac{i}{2N}\right| \le F_p; \quad F_p \le \frac{1}{2N} \\ 0 & \left|f - \frac{i}{2N}\right| \in (F_p, F_s] \\ & \text{or } \left|f + \frac{i}{2N}\right| \in (F_p, F_s]; \quad F_s \le \frac{1}{N}{}^{1} \\ w & \text{elsewhere}^{2}, \end{cases} \quad \text{(25b)}$$

where $i = 0, 2, 4, \cdots$, for structure (A), $i = 1, 3, 5,$

[1]For $F_s > 1/2N$ this definition becomes inaccurate for $i = 1$ [the first filter in structure (B)]. To accommodate for this particular case in the design process, (25)-(29) have to be slightly modified. However, this modification complicates the presentation and therefore is omitted here.

$\cdots$, for structure (B), and $C_i \triangleq 1$ except for $C_o = C_N \triangleq 0.5$.

Fig. 5(a) illustrates a typical weight function $|W_i(f)|^2$ and Fig. 5(b) illustrates the equivalent low-pass weight function $|W_o(f)|^2$ used in the corresponding CUDP.

The real filter banks obtained from a PTFB have the following property:

$$H_i(f) = C_i\left[H_o\left(f + \frac{i}{2N}\right) + H_o\left(f - \frac{i}{2N}\right)\right], \quad (26)$$

where the values of i in (26) are determined by the structure used in combining pairs of outputs of the PTFB. In general, the optimal solution of the RUDP does not possess the property implied by (26). Thus, unlike the CUDP case, in general, any PTFB provides only a *suboptimal* solution of the RUDP. However, one can impose (26) as part of the design specifications to guarantee an efficient implementation via a PTFB. In this context, the question of finding the *optimal prototype filter* arises naturally. We concentrate, therefore, on the optimal (in the WMMSE sense) solution of the RUDP subject to the constraint imposed by (26), with the Euclidean norm used as $\| \, \|_T$.

For this case, the composite response specification imposed on the prototype filter is given by (11b) as in the WMMSE CUDP (with $\Delta = 0$), whereas the overall error [which is obtained by substituting (26), (25), and (10) in (2), (3), and (5)] is, for structure (A),

$$\epsilon_A^2 = \begin{cases} \epsilon_{cx}^2 + N\psi_N; & N \text{ odd} \\ \epsilon_{cx}^2 + N\psi_{N/2}; & N \text{ even} \end{cases} \quad (27a)$$

and for structure (B),

$$\epsilon_B^2 = \begin{cases} \epsilon_{cx}^2 + N\psi_N; & N \text{ odd} \\ \epsilon_{cx}^2 + N(2\psi_N - \psi_{N/2}); & N \text{ even} \end{cases} \quad (27b)$$

where ϵ_{cx}^2 is the overall error of the WMMSE CUDP given in (11a), and ψ_N is the following correction term:

$$\psi_N \triangleq \frac{1}{N}\sum_{i=1}^{N-1}\left\{\int_{-0.5}^{0.5}\left(|W_o(f)|^2 - w\right)\left|D_o\left(f + \frac{i}{N}\right) - H_o\left(f + \frac{i}{N}\right)\right|^2 df + \int_{-0.5}^{0.5}\left(2|W_o(f)|^2 - w\right)\operatorname{Re}\left[\left(\overline{D_o(f)} - \overline{H_o(f)}\right)\left(D_o\left(f + \frac{i}{N}\right) - H_o\left(f + \frac{i}{N}\right)\right)\right] df\right\} \quad (27c)$$

where the "superbar" denotes complex conjugation. Due to the existence of a nonzero correction term ψ_N, ϵ_A^2 and ϵ_B^2 differ from the overall error ϵ_{cx}^2 of the WMMSE CUDP. Thus, the optimal prototype for the constrained WMMSE RUDP is *different* from the optimal prototype for the WMMSE CUDP. Furthermore, for an even number of filters, the optimal prototype for structure (A) is not the optimal prototype for structure (B). For optimization purposes, the correction term ψ_N is rewritten as the following quadratic form:

$$\psi_N = \tfrac{1}{2}\boldsymbol{a}_o^*(R_N^* + R_N)\boldsymbol{a}_o - \boldsymbol{d}_N^*\boldsymbol{a}_o - \boldsymbol{a}_o^*\boldsymbol{d}_N + \psi_o \quad (28)$$

where ψ_o is a constant (independent of $\boldsymbol{a}_o$), and the elements of the $M \times M$ matrix R_N are

$$R_N(m,k) = -\frac{1}{N}\left(3R_o(m,k) - 2w\right) + \left(R_o(m,k) - w\right)\delta(m \equiv k \bmod N) + \left(2R_o(m,k) - w\right)\delta(k \equiv 0 \bmod N) \quad (29a)$$

and the elements of $\boldsymbol{d}_N \in \mathbb{C}^M$ (assuming that $D_o(f) = 0$ for $|f| > 1/2N$) are

$$d_N(m) = -\frac{1}{N}d_o(m) + d_o(m)\,\delta(m \equiv 0 \bmod N); \quad m = 0, \cdots, M-1. \quad (29b)$$

Therefore, the design of the optimal prototype for the constrained WMMSE RUDP is via (13a)–(13g) with R_o and $\boldsymbol{d}_o$ being replaced by the following expressions, respectively:

(a) $R_o + \frac{1}{2}(R_N + R_N^*)$; $\boldsymbol{d}_o + \boldsymbol{d}_N$ for N odd

(b) $R_o + \frac{1}{2}(R_{N/2} + R_{N/2}^*)$; $\boldsymbol{d}_o + \boldsymbol{d}_{N/2}$ for N even and structure (A)

(c) $R_o + R_N + R_N^* - \frac{1}{2}(R_{N/2} + R_{N/2}^*)$; $\boldsymbol{d}_o + 2\boldsymbol{d}_N - \boldsymbol{d}_{N/2}$ for N even and structure (B).

The issue of design complexity is similar to the CUDP case, and was already discussed in Section III.

VII. Conclusions

Three new algorithms for solving a class of filter bank design problems denoted as complex uniform design problems (CUDP) are presented. They are based on the existence of an optimal solution which is a prototype translated filter bank (PTFB). This important existence theorem (Theorem 1) results in an equivalent design problem with reduced complexity.

The first algorithm is a simplified version of the general WMMSE design method presented in [1], and it typically reduces the complexity of the design by two orders of magnitudes.

The second algorithm is for min–max CUDP. It is based on linear programming (Section IV) of reduced dimen-

[2] $W_i(f)$ is periodic with a period of 1, and in that context we interpret the word "elsewhere."

sionality, and also accommodates tolerance specifications on the composite response. For the particularly important case of a flat composite response specification with zero tolerance value, this algorithm is further simplified and coincides with the Nyquist filter design method of [10].

Nevertheless, the need to use linear programming techniques practically sets a limit on the length of the filters, due to the high design complexity. To overcome this obstacle, the relationship between an important subclass of CUDP and the window method for FIR filter design is pointed out. It is shown that the optimal PTFB for flat composite response specification with zero tolerance value can be obtained by the window method provided that an optimal window is used. An approximate optimal window (AOW) is derived following the analysis of [11] and [15], and its design by the Remez exchange algorithm using an available program [4] is presented. The third algorithm is based on the AOW sequence. It is a suboptimal algorithm for min-max CUDP, but it results in a PTFB which is only slightly degraded with respect to the optimal solution, and this design method is of reduced complexity. Thus, it is suitable for CUDP with relatively long FIR filters (typically several hundred taps). Furthermore, for nonuniform filter banks, the AOW sequence is expected to lead to superior designs over the conventional window method.

The outputs of a PTFB are complex valued signals, and what is commonly done when real valued outputs are desired is the addition of proper pairs of outputs of the PTFB yielding a real filter bank (cf. [3]). Unlike the CUDP for which there always exists a solution which is a PTFB, imposing this filter bank structure usually results in a *suboptimal solution* for the corresponding real uniform design problem (RUDP). However, for applications in which this structure is imposed as part of the design specifications, the (simplified) design of the optimal (in the WMMSE sense) prototype filter is presented. It is shown that *this optimal prototype filter differs from the solution of the WMMSE CUDP*.

Although the design of Nth band filters, Nyquist filters, or partial response filters [10], [12] are not in the scope of this paper, we point out that the equivalence between these problems and the CUDP enables the use of the three new algorithms for the design of these filters as well.

Appendix

Proof of Theorem 1: There exists at least one optimal solution, $\{a_{ik}\}_{i=1,k=-L}^{N,L}$ (as shown in [16]). Let its frequency responses be denoted by $\{H_i(f)\}_{i=1}^N$. Define $\hat{H}_i(f) \triangleq 1/N \sum_{l=1}^N H_l(f-(l-i)/N)$, for $i = 1, \cdots, N$. It is easy to verify that $\hat{H}_i(f)$ is the frequency response of the ith filter in a PTFB. Using the triangle inequality, and homogeneity of $\| \ \|$ together with (7a), (7c), and (7f) results in $\hat{\delta}_i \triangleq \| W_i(f)(D_i(f) - \hat{H}_i(f))\| \le 1/N \sum_{l=1}^N \| W_o(f)(D_o(f) - H_l(f - \Delta - l/N))\|$. Using (7a), (7c), and (7f) transfers the above inequality into: $\hat{\delta}_i \le 1/N \sum_{l=1}^N \delta_l$. The "monotonicity" of $\| \ \|_T$, together with the triangle inequality and homogeneity of $\| \ \|_T$, results in the following inequality: $\hat{\epsilon} \triangleq \| \hat{\boldsymbol{\delta}} \|_T \le \| 1/N \sum_{l=1}^N \boldsymbol{\delta}^{(l)} \|_T \le 1/N \sum_{l=1}^N \| \boldsymbol{\delta}^{(l)} \|_T$. However, from (7e) this is equivalent to $\hat{\epsilon} \le \epsilon$. Thus, the PTFB achieves the minimal error, and it only remains to show that $\hat{\delta}_c \triangleq \| W_c(f)(D_c(f) - \hat{H}_c(f))\| \le \delta_c$. Rearranging the expression for the composite response of the PTBF as $\hat{H}_c(f) = 1/N \sum_{j=1}^N H_c(f - j/N)$, using the triangle inequality and homogeneity of $\| \ \|$, results in $\hat{\delta}_c \le 1/N \sum_{j=1}^N \| W_c(f)(D_c(f) - H_c(f - j/N))\|$. From (7b), (7d), and (7f) follows that $\hat{\delta}_c \le \delta_c$, and the proof is completed. ■

Proof of Theorem 2: Equation (9) follows from the definition of the PTFB, and equation (1), where $\{a_{ok}\}_{k=-L}^L$ are the coefficients of the prototype FIR filter with the frequency response $H_o(f)$. From (2), (3), (7a), (7c), and (7f) it follows that in any PTFB for a CUDP all the N components of $\boldsymbol{\delta}$ are equal. Thus, due to the "monotonicity" of $\| \ \|_T$, the optimal prototype minimizes each component of $\boldsymbol{\delta}$, i.e., it is the solution of (8a). Equation (8b) follows from the definition of the PTFB, (4), (7f), and the well-known result

$$\sum_{i=0}^{N-1} e^{-j2\pi ik/N} = \begin{cases} N & k \equiv 0 (\mathrm{mod}\ N) \\ 0 & \text{elsewhere} \end{cases}$$

which is applicable since L is an integer value. Since $a_{ik} = 0$ for $|k| > L$, so is a_{ok}, and therefore, in (8b), $|mN| \le L$. ■

Proof of Theorem 3: In the window method $H_i(f) = D_i(f) * W(f)$, with $W(f)$ being the frequency response of the window sequence. Using (7a) and elementary properties of the convolution operator leads to $H_i(f) = [D_o * W](f + i/N + \Delta) \triangleq H_o(f + i/N + \Delta)$, i.e., the window method always results in a PTFB. The composite response of this PTFB is

$$H_c(f) = \sum_{i=1}^N D_i(f) * W(f) = \left[\sum_{i=1}^N D_i(f) \right] * W(f)$$
$$= 1 * W(f) = w_o \quad (\text{cf. } [8]),$$

where we have used the odd length of the window in the last equation. Thus, for any window with $w_o = 1$. The window method results in a PTFB with $\eta = \eta_m = 0$. For $D_o(f)$ which is an ideal LPF of bandwidth $1/2N$, the coefficients of the corresponding impulse response $d_o(k)$ are zero *only* for $k \equiv 0(\mathrm{mod}\ N)$, and $d_o(o) = 1/N$. Therefore, any PTFB with a flat composite response can be designed by the window method, provided that the proper window sequence is used (i.e., $w_k = a_{ok}/d_o(k)$, $k \not\equiv 0(\mathrm{mod}\ N)$, $w_o = 1$). In particular, the optimal PTFB can be obtained in this manner by the window method. ■

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REFERENCES

[1] A. Dembo and D. Malah, "WMMSE design of digital filter banks with specified composite response," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 1529–1542, Dec. 1986.
[2] B. A. Doutrich, L. R. Rabiner, and T. B. Martin, "On the effects of varying filter bank parameters on isolated word recognition," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 793–807, Aug. 1983.
[3] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing*, Signal Processing Series. Englewood Cliffs, NJ: Prentice-Hall, 1983.
[4] J. H. McClellan, T. W. Parks, and L. R. Rabiner, "A computer program for designing optimum FIR linear phase digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 506–526, Dec. 1973.
[5] D. C. Farden and L. L. Scharf, "Statistical design of non-recursive digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-22, pp. 188–196, June 1974.
[6] A. Dembo and D. Malah, "Design of digital filter banks with flat composite response," in *Conf. Digital Signal Processing*, Florence, Italy, 1984, pp. 22–26.
[7] Y. C. Lim, "Efficient special purpose linear programming for FIR filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, Aug. 1983.
[8] R. W. Schafer, L. R. Rabiner, and O. Herrman, "FIR digital filter banks for speech analysis," *BSTJ*, vol. 54, no. 3, pp. 531–544, Mar. 1975.
[9] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975, ch. 3.
[10] J. K. Liang, R. J. P. de Figueiredo, and F. C. Lu, "Design of optimal Nyquist, partial response, *N*th band, and nonuniform tap spacing FIR digital filters using linear programming techniques," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 386–391, Apr. 1985.
[11] A. Dembo and D. Malah, "Generalization of the window method for FIR digital filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 1081–1083, Oct. 1984.
[12] F. Mintzer, "On half-band, third-band, and *N*th band FIR filters and their design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 734–738, Oct. 1982.
[13] M. Avriel, *Nonlinear Programming: Analysis and Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1976.
[14] L. R. Rabiner, "Linear program design of finite impulse response (FIR) digital filters," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 401–408, July 1973.
[15] A. Dembo and D. Malah, "Ganeralization of the window method for FIR digital filter design," Technion—Israel Inst. Technol., Haifa, Israel, EE. Pub. No. 456, July 1983.
[16] A. Dembo, "The design of digital filter arrays," D.Sc. dissertation, Appendix B; in Hebrew, Technion—Israel Inst. Technol., Haifa, Israel, May 1986.

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