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DFT INTERPOLATION KERNELS AND ERROR BOUNDS

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Abstract—The interpolation formula representation and the kernels associated with the Discrete Fourier Transform (DFT) approach to the interpolation of periodic signals are obtained by viewing the interpolation process as a filtering operation on a properly defined sequence. This representation provides then the basis for the derivation of upper bounds on the interpolation error involved.

INTRODUCTION

Let $f(t)$ be a continuous periodic real signal, with period T_0, and let $g(t)$ be the steady state response of a linear stable system R to the signal $f(t)$. Given N equally spaced samples of one period of $f(t)$, the problem under consideration is that of finding NL equally spaced samples of one period of $g(t)$, where $L \geq 1$ is an integer. A simple and computationally efficient method for performing the desired interpolation is to apply the Discrete Fourier Transform (DFT), implemented with the FFT algorithm. This method has been considered and discussed (for the interpolation of $f(t)$) in Ref.[1], and is outlined briefly in Refs[2, 3]. One of the aims of this presentation is the derivation of upper bounds on the interpolation error associated with the DFT approach. The derivation is based on an interpolation formula representation of the interpolation process. The interpolation kernels which appear in the interpolation formula are derived in a simple manner, by approaching the interpolation problem as a filtering operation on a properly defined sequence which is constructed from the given samples. In the following sections the DFT interpolation procedure is described, interpolation kernels are derived, and, finally, upper bounds on the interpolation error are found.

DFT INTERPOLATION KERNELS

Let $C(k)$ and $B(k)$, $k = 0, \pm 1, \pm 2, \ldots$, be the Fourier Series coefficients of the periodic functions $f(t)$ and $g(t)$, respectively. Then, $B(k) = C(k)R(k)$, where $R(k) = R(j\omega_0 k)$ is given from the frequency response $R(j\omega)$ of the system R and $\omega_0 = 2\pi/T_0$.

Denoting the given sequence of N samples of $f(t)$ by $x(n)$, $n = 0, 1, \ldots, N-1$, and the desired interpolated NL samples of $g(t)$ by $y_L(n) = g(nT/L)$, $n = 0, 1, \ldots, NL-1$, where $T = T_0/N$, the following relations are known[4] to exist between $X(k) \triangleq \mathrm{DFT}\{x(n)\}$ and $C(k)$, and between $Y_L(k) \triangleq \mathrm{DFT}\{y_L(n)\}$ and $B(k)$.

$$X(k) = \sum_{r=-\infty}^{\infty} C(k + rN); \quad Y_L(k) = \sum_{r=-\infty}^{\infty} B(k + rNL). \tag{1}$$

The DFT of the sequence $x(n)$ is defined in eqn (2), where $W_N \triangleq \exp(j2\pi/N)$.

$$X(k) = \mathrm{DFT}\{x(n)\} \triangleq \frac{1}{N} \sum_{n=0}^{N-1} x(n) W_N^{-nk}, \quad k = 0, 1, \ldots, N-1 \tag{2}$$

$Y_L(k)$ is similarly defined (with N replaced by NL).

If $f(t)$ is base-band limited, i.e. $C(k) = 0$ for $k > M$, and $N \geq 2M + 1$, $Y_L(k)$ can be constructed from $X(k)$ and $R(k)$ as shown in eqn (3).

$$\begin{gathered} Y_L(k) = X(k)R(k); \quad Y_L(NL - k) = X(N - k)R(-k), \quad k = 0, 1, \ldots, M; \\ Y_L(k) = 0, \quad k = M + 1, \ldots, NL - M - 1. \end{gathered} \tag{3}$$

Now that $Y_L(k)$ is known, $y_L(k)$ is obtained by inverse transforming $Y_L(k)$:

$$y_L(n) = \mathrm{DFT}^{-1}\{Y_L(k)\} = \sum_{k=0}^{N-1} Y_L(k) W_{NL}^{nk}. \tag{4}$$

For the particular case in which $R(j\omega)=1$, the interpolation of the input signal $f(t)$ is obtained and $y_L(n)=x_L(n)\stackrel{\Delta}{=}f(nT/L)$.

The DFT interpolation procedure is therefore as follows: Find $X(k)$ from the given sequence $x(n)$, construct $Y_L(k)$ as in eqn (3), and then apply the inverse transform defined in eqn (4). The same procedure is also applied when $f(t)$ is not band-limited but with the following modifications:

(i) If N is odd, use in eqn (3) $M=(N-1)/2$.
(ii) If N is even, M is given the value $N/2$, and in addition one should use in eqn (3):

$$Y_L(N/2)=X(N/2)R(N/2)/2;\quad Y_L(NL-N/2)=X(N/2)R(-N/2). \tag{5}$$

This last modification, for N even, takes into account the overlap of $B(N/2)$ and $B(-N/2)$ which occurs at $k=N/2$; (note that $B(N/2)=B^*(-N/2)$). Inverse transforming the resulting sequence, which we denote by $\hat{Y}_L(k)$, we arrive at a band-limited approximation $\hat{y}_L(n)$ to the desired sequence $y_L(n)$.

We turn now to the interpolation formula representation which is of the form

$$\hat{y}_L(n)=\sum_{r=0}^{N-1}x(r)h_d(n-rL),\quad n=0,1,\ldots,NL-1, \tag{6}$$

where $h_d(n)$ is an appropriate interpolation kernel. The interpolation kernel can be found by properly manipulating the expressions given above. However, by viewing the interpolation process as a filtering operation on the sequence $s(n)$ defined in eqn (7) below, the kernel derivation is simplified.

The sequence $s(n)$ discussed above is of length NL and is constructed from $x(n)$ by inserting $(L-1)$ zeroes between every two samples of $x(n)$.

$$s(n)=\text{Stretch}_L\{x(n)\}\stackrel{\Delta}{=}\{x(0)0\ldots0x(1)0\ldots0x(N-1)0\ldots0\}. \tag{7}$$

Using Theorem 8 in Refs. [4, 5] we have

$$S(k)\stackrel{\Delta}{=}\text{DFT}\{s(n)\}=X(k)/L,\quad k=0,1,\ldots,NL-1. \tag{8}$$

Since $X(k)$ is periodic, with period N, $S(k)$ contains L such periods. $Y_L(k)$ as given in eqn (3) can therefore be constructed by multiplying $S(k)$ by a proper sequence $H(k)$ (i.e. filtering $s(n)$). For the non-band-limited case, which is of interest here, the required sequence $H(k)$ is given by $H_o(k)$ for N odd and by $H_e(k)$ for N even, where

$$\begin{gathered}H_o(k)=LR(k);\quad H_o(NL-k)=LR(-k),\quad k=0,1,\ldots,(N-1)/2\\ H_o(k)=0,\quad k=(N+1)/2,\ldots,NL-(N+1)/2\end{gathered} \tag{9}$$

$$\begin{gathered}H_e(k)=LR(k);\quad H_e(NL-k)=LR(-k),\quad k=0,1,\ldots,(N/2)-1\\ H_e(N/2)=LR(N/2)/2;\quad H_e(NL-N/2)=LR(-N/2)/2\\ H_e(k)=0,\quad k=(N/2)+1,\ldots,NL-(N/2)+1.\end{gathered} \tag{10}$$

Applying the DFT Convolution Theorem[5], we obtain

$$\hat{y}_L(n)\stackrel{\Delta}{=}\text{DFT}^{-1}\{\hat{Y}_L(k)\}=\text{DFT}^{-1}\{H(k)S(k)\}=\frac{1}{NL}\sum_{r=0}^{NL-1}s(r)h(n-r) \tag{11}$$

where

$$h(n) \triangleq \mathrm{DFT}^{-1}\{H(k)\}.$$

Using the definition of $s(n)$, according to eqn (7), in eqn (11), the desired interpolation formula (6) is obtained with

$$h_d(n) \triangleq h(n)/NL = \mathrm{DFT}^{-1}\{H(k)\}/NL. \tag{12}$$

For illustration we present the interpolation kernels obtained for two particular cases: (i) $R(j\omega) = 1$ [interpolation of $f(t)$], in which case we found

$$\begin{aligned} h_d(n) &= [\sin(\pi n/L)]/N \sin(\pi n/NL), \quad N \text{ odd} \\ &= [\sin(\pi n/L)\cot(\pi n/NL)]/N, \quad N \text{ even}. \end{aligned} \tag{13}$$

(ii) $R(j\omega) = -j\,\mathrm{sgn}(\omega)$ [Hilbert Transform], in which case we found

$$\begin{aligned} h_d(n) &= [\cot(\pi n/NL)]/N - [\cos(\pi n/L)]/N \sin(\pi n/NL), \quad N \text{ odd} \\ &= [\cot[\pi n/NL)][1-\cos(\pi n/L)]/N, \quad N \text{ even}. \end{aligned} \tag{14}$$

INTERPOLATION ERROR BOUNDS

The interpolation error under consideration is defined by

$$e(n) \triangleq y_L(n) - \hat{y}_L(n), \quad n = 0, 1, \ldots, NL-1. \tag{15}$$

It is shown in the Appendix that

$$|e(n)| \le \epsilon \triangleq 2 \sum_{k=N_0}^{\infty} [|B(k)| + |\tilde{B}(k)|] \tag{16}$$

where $N_0 = (N+1)/2$ for N odd, and $N_0 = N/2$ for N even, $B(k) = C(k)R(k)$ and

$$\tilde{B}(k) \triangleq C(k)R(\tilde{k}); \quad \tilde{k} \triangleq k \bmod N_0, \quad 0 \le \tilde{k} < N_0. \tag{17}$$

For the two particular cases $R(j\omega) = 1$ and $R(j\omega) = -j\,\mathrm{sgn}(\omega)$ considered above, ϵ becomes equal to ϵ_I, where

$$\epsilon_I \triangleq 4 \sum_{k=N_0}^{\infty} |C(k)|. \tag{18}$$

The upper bound in (18) complements the bound obtained in Ref. [6] for non-periodic interpolation. Such an upper bound can be useful for choosing the sampling rate of a given function $f(t)$. It is not possible, however, to evaluate ϵ from eqn (18) if we are given only N samples of $f(t)$. Yet, in some applications there is additional knowledge in the physical process which generated $f(t)$, so that upper bounds on the first or higher derivatives of $f(t)$ can be estimated. The bounds on the derivatives can then be used to bound ϵ as shown below (Part of the derivation has been motivated by the work in Ref. [7]).

Let D_p be an upper bound on the p derivative of $f(t)$: $D_p \ge |f^{(p)}(t)|$ (with $f^{(0)}(t) \triangleq f(t)$). Noting that $B(k)$ can be written as

$$B(k) \triangleq C(k)R(k) = [C(k)k^p\omega_0^{\,p}][R(k)/k^p\omega_0^{\,p}] \tag{19}$$

and applying the Schwarz inequality, we obtain

$$\epsilon_1 \triangleq \sum_{k=N_0}^{\infty} |B(k)| \leq \left[\sum_{k=N_0}^{\infty} |C(k)k^p\omega_0{}^p|^2 \sum_{k=N_0}^{\infty} |R(k)/k^p\omega_0{}^p|^2\right]^{1/2}. \tag{20}$$

Observing now that the Fourier series coefficients of $f^{(p)}(t)$, which we denote by $C_p(k)$, are related to $C(k)$ by $C_p(k) = (j\omega_0 k)^p C(k)$, and further, applying the well known Fourier series equality

$$\sum_{k=-\infty}^{\infty} |C_p(k)|^2 = \frac{1}{T_0}\int_0^{T_0} [f^{(p)}(t)]^2 \, dt \leq D_p{}^2 \tag{21}$$

as well as $|C(-k)| = |C(k)|$ ($f(t)$ is real), we find

$$\epsilon_1 \leq (D_p/\sqrt{(2)})\left[\sum_{k=N_0}^{\infty} |R(k)/k^p\omega_0{}^p|^2\right]^{1/2} \triangleq I_1. \tag{22}$$

When $R(j\omega)$ is specified, p is chosen so that the series in eqn (22) converges and I_1 can then be evaluated. Consider for example a specification of the form

$$R(j\omega) = (j\omega)^m, \quad m \geq 0. \tag{23}$$

For this specification, letting $p = m + 1$ assures the convergence of the series in eqn (22), and we obtain that I_1 is equal to $I_1(m)$ where

$$I_1(m) \triangleq D_{m+1} Z(N_0)/\sqrt{2}\omega_0 \tag{24}$$

and where

$$Z(N_0) \triangleq \left[\sum_{k=N_0}^{\infty} 1/k^2\right]^{1/2} = \left[\pi^2/6 - \sum_{k=1}^{N_0-1} 1/k^2\right]^{1/2}. \tag{25}$$

A computation of $Z(N_0)$ as function of N_0 shows that it behaves like $1/\sqrt{(N_0)}$ for $N_0 \gg 1$. [$Z(N_0)$ is within 5% of $1/\sqrt{(N_0)}$ for $N_0 \geq 5$ and within 1% for $N_0 \geq 20$.]

Note that the case $m = 0$, ($R(j\omega) = 1$) provides an upper bound for ϵ_I of eqn (18):

$$\epsilon_I \leq 4I_1(0) = 2\sqrt{(2)} D_1 Z(N_0)/\omega_0. \tag{26}$$

To obtain an upper bound on ϵ of eqn (16), we still have to consider $\Sigma\tilde{B}(k)$. In general, we can use:

$$\sum_{k=N_0}^{\infty} |\tilde{B}(k)| \triangleq \epsilon_2 \leq M_R \sum_{k=N_0}^{\infty} |C(k)| \leq M_R I_1(0) \tag{27}$$

where

$$M_R \triangleq \max_{k\in[0,\, N_0)} \{|R(k)|\}. \tag{28}$$

Hence we obtain, from eqns (16), (22) and (27)

$$\epsilon = 2(\epsilon_1 + \epsilon_2) \leq 2[I_1 + M_R I_1(0)]. \tag{29}$$

In general, M_R can be a function of N_0. If M_R increases with N_0, the bound in (29) may not be useful. However, if $R(k) \geq R(N_0)$ for all $k > N_0$, as is the case for the example specified in eqn (23), then we can use $\epsilon_1 \geq \epsilon_2$ and obtain

$$\epsilon \leq 4\epsilon_1 \leq 4I_1. \tag{30}$$

Finally, for the specification in eqn (23), we find from eqns (30) and (24),

$$\epsilon \leq 4I_1(m) = 2\sqrt{(2)}D_{m+1}Z(N_0)/\omega_0 \tag{31}$$

which reduces to eqn (26) for $m = 0$.

CONCLUSIONS

In most practical applications the interpolation process is most efficiently performed by the DFT procedure described above. However, this procedure is of algorithmic nature and does not provide the proper framework for further investigations. On the other hand, the interpolation formula representation, with the associated kernels, provide a complete and explicit description of the interpolation process and has here been found useful for investigating the interpolation error involved.

The filtering point of view taken to describe the interpolation process provides a simple method for the derivation of interpolation kernels. The error bounds obtained can provide additional insight into practical interpolation problems.

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APPENDIX

The upper bound, on the interpolation error sequence given in eqn (16), is found as follows:

$$e(n) \triangleq y_L(n) - \hat{y}_L(n) = \sum_{k=-\infty}^{\infty} B(k)\exp[j2\pi kn/NL] - \sum_{r=0}^{N-1} x(r)h_d(n-rL). \tag{A1}$$

Using

$$x(r) = \sum_{k=-\infty}^{\infty} C(k)\exp[j2\pi kr/N], \quad r = 0, 1, \ldots, N-1 \tag{A2}$$

we find

$$e(n) = \sum_{k=-\infty}^{\infty} C(k)[R(k)\exp(j2\pi kn/NL) - V_k(n)] \tag{A3}$$

where

$$V_k(n) = \sum_{r=0}^{N-1} [\exp(j2\pi kr/N)]h_d(n-rL). \tag{A4}$$

Recalling that $h_d(n)$ is a kernel which yields exact interpolation for band-limited periodic signals, which pass through the system R, we obtain

$$\begin{aligned} V_k(n) &= R(k)\exp[j2\pi kn/NL], \quad |k| \leq N_o - 1 \\ &= R(\bar{k})\exp[j2\pi\bar{k}n/NL], \quad |k| \geq N_o \end{aligned} \tag{A5}$$

except that for N even $V_{N_0}(n) = R(N_0)\cos(\pi n/L)$. From eqns (A3) and (A5) we find

$$e(n) = \sum_{|k| \geq N_1} [B(k)\exp[j2\pi kn/NL] - \bar{B}(k)\exp[j2\pi\bar{k}n/NL]] + B_0(n) \tag{A6}$$

where, for N odd, $N_1 = N_0 = (N+1)/2$ and $B_0(n) = 0$, whereas for N even, $N_1 = N_0 - 1 = N/2 - 1$, and

$$B_0(n) = j[B(N/2) - B(-N/2)]\sin(\pi n/L), N \text{ even}. \tag{A7}$$

Now, since $|B(N/2)| = |B(-N/2)|$ and $|B_0(n)| < 2|B(N/2)|$, we finally obtain the result in eqn (16). Note that, for N even, eqns (A6) and (A7) can actually yield a somewhat better bound than ϵ of eqn (16), by using $|B_0(n)| \leq 2|Im\{B(N/2)\}|$.