CASCADE DECOMPOSITION OF LINEAR TIME-VARYING DIFFERENCE OPERATORS

Indexing terms: Time-varying systems, Difference equations

Two methods are given for the cascade decomposition of a scalar time-varying difference operator, of order m, under the assumption that m linearly independent solutions of the homogeneous equation are known. The relationship between the two decomposition schemes obtained by the two methods is derived, and an example is given.

Introduction: Certain discrete time-varying systems can be described by a scalar difference equation of the form

$$y(n+m)+a_1(n)\,y(n+m-1)+\ldots+a_m(n)\,y(n) = u(n) \qquad (1)$$

where $u(n)$ is the forcing function, $y(n)$ is the output, and m is the order of the system $\{a_m(n) \neq 0\}$. In operational notation, eqn. 1 can be written as $L_m(n)\,y(n) = u(n)$, where

$$L_m(n) = \sum_{i=0}^{m} a_i(n)\,E^{m-i} \qquad a_0(n) = 1 \quad . \quad . \quad . \quad (2)$$

in which E is the advance operator; i.e. $Ey(n) = y(n+1)$.

For time-invariant systems, $L_m(n) = L_m$, and the z transform can be applied for deriving a cascade decomposition of L_m. For time-varying systems, the use of transform methods† is limited and, in general, is not simple. A time-domain approach is therefore chosen, and two decomposition methods are presented.

Decomposition method 1: The decomposition procedure is based on the reduction of order when a solution of $L_m(n)\,y(n) = 0$ is known, as shown by Milne-Thomson.[2] Consequently, it is assumed in this discussion that a set of m linearly independent solutions $\{f(n)\}_m$ of the homogeneous equation $L_m(n)\,y(n) = 0$ is known. Such is the case when the impulse-response function $h(n, k)$ is known.‡

† For example, the use of the generalised system function[1] $H(z, n)$

‡ The response to δ_{nk} (Kronecker delta) is known[3] to have the form

$$h(n, k) = \sum_{i=1}^{m} f_i(n)\,g_i(k)$$

where $f_i(n)$, $i = 1, 2, \ldots, m$, satisfy the homogeneous equation $L_m\,y(n) = 0$

According to Milne-Thomson, if $f_1(n)$ is a solution of $L_m(n)\,y(n) = 0$, a difference equation of order $m-1$

$$L_{m-1}(n)\,w_1(n) = u(n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

can be derived. The relationship between $w_1(n)$ of eqn. 3 and $y(n)$ of eqn. 1 is given from

$$y(n) = f_1(n)\,v_1(n) \qquad \Delta v_1(n) = w_1(n) \quad . \quad . \quad . \quad (4)$$

where Δ is the forward-difference operator; i.e.

$$\Delta v_1(n) = v_1(n+1) - v_1(n)$$

Thus, if Δ^{-1} denotes the inverse of Δ,

$$y(n) = f_1(n)\,\Delta^{-1}\,w_1(n)$$

The above corresponds to the realisation of $L_m(n)$ by $L_{m-1}(n)$ of eqn. 3 in cascade with a 1st-order operator $L_1{}^1(n)$ such that

$$L_1{}^1(n)\,y(n) = w_1(n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From eqns. 4 and 5, the schematic representation of $L_1{}^1(n)$ shown in Fig. 1 is obtained.

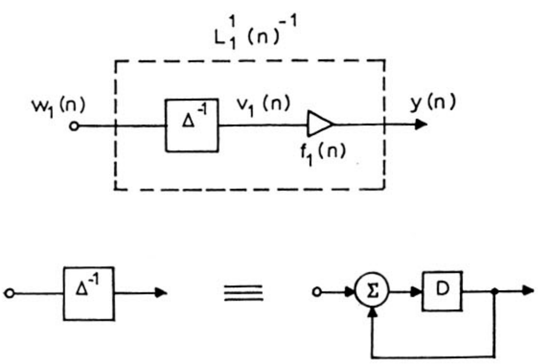

Fig. 1 *Representation of the 1st-order operator $L_1{}^1(n)$*
D represents a unit delay

A set of $(m-1)$ linearly independent functions which satisfies $L_{m-1}(n)\,w_1(n) = 0$ is obtained from $\{f(n)\}_m$ by[2]

$$f_i{}^1(n) = \Delta \frac{f_i(n)}{f_1(n)} \qquad i = 2, 3, \ldots, m \quad . \quad . \quad . \quad . \quad (6)$$

Since $\{f(n)\}_m$ is assumed to be known, the functions $f_i{}^1(n)$, $i = 2, 3, \ldots, m$, can be derived. The reduction of $L_{m-1}(n)$ can now be undertaken in a similar way by using a solution of $L_{m-1}(n)\,w_1(n) = 0$, say $f_2{}^1(n)$, and decomposing $L_{m-1}(n)$ into $L_{m-2}(n)\,L_1{}^2(n)$, so that $L_{m-2}(n)\,w_2(n) = u(n)$ and $w_2(n)$ is related to $w_1(n)$ by

$$w_1(n) = f_2{}^1(n)\,v_2(n) \qquad \Delta v_2(n) = w_2(n) \quad . \quad . \quad . \quad (7)$$

A set of $m-2$ linearly independent solutions of

$$L_{m-2}(n)\,w_2(n) = 0$$

is found from

$$f_i{}^2(n) = \Delta \frac{f_i{}^1(n)}{f_2{}^1(n)} \qquad i = 3, 4, \ldots, m \quad . \quad . \quad . \quad . \quad (8)$$

Repetition of the above procedure would finally yield the decomposition of $L_m(n)$ into m 1st-order operators,

$$L_m(n) = L_1{}^m(n)\,L_1{}^{m-1}(n) \ldots L_1{}^1(n)$$

which corresponds to the schematic representation shown in Fig. 2, in which

$$r_1(n) = f_1(n) \qquad r_i(n) = f_i{}^{i-1}(n) \qquad i = 2, 3, \ldots, m \qquad (9)$$

By repeated use of eqn. 6, one obtains

$$r_i(n) = \Delta\left[\frac{1}{r_{i-1}(n)}\left\{\Delta \frac{1}{r_{i-2}(n)} \cdots \frac{1}{r_2(n)}\,\Delta \frac{f_i(n)}{r_1(n)}\right\}\right] \qquad i = 2, 3, \ldots, m \quad (10)$$

It is also evident from Fig. 2 that

$$L_m(n)\,y(n) = \frac{1}{r_{m+1}(n)}\,\Delta\left[\frac{1}{r_m(n)}\left\{\Delta \frac{1}{r_{m-1}(n)} \cdots \frac{1}{r_2(n)}\,\Delta \frac{y(n)}{r_1(n)}\right\}\right] = u(n) \qquad (11)$$

where $r_{m+1}(n)$ was introduced to ensure $a_0(n) = 1$, as assumed

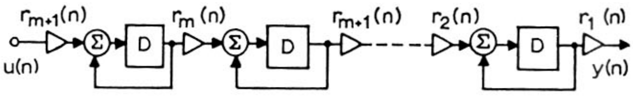

Fig. 2 *Cascade decomposition of $L_m(n)\,y(n) = u(n)$*

in eqn. 2. $r_{m+1}(n)$ must therefore satisfy

$$\frac{1}{r_{m+1}(n)} = r_m(n+1)\,r_{m-1}(n+2) \ldots r_2(n+m-1)\,r_1(n+m) \quad (12)$$

It is easily verified, by use of eqns. 9 and 10, that eqns. 11 do satisfy $L_m f_i(n) = 0$, $i = 1, 2, \ldots, m$.

Decomposition method 2: Here, $L_m(n)$ is decomposed as

$$L_m(n) = Q_m(n)\,Q_{m-1}(n) \ldots Q_1(n) \quad . \quad . \quad . \quad . \quad . \quad (13a)$$

where

$$Q_i(n) = E - q_i(n) \qquad i = 1, 2, \ldots, m \quad . \quad . \quad . \quad . \quad (13b)$$

and E is the advance operator. The corresponding schematic representation is shown in Fig. 3.

To satisfy $L_m(n) f_i(n) = 0$, $i = 1, 2, \ldots, m$, the $Q_i(n)$s are chosen so that

$$\left.\begin{aligned} &Q_1(n)\,f_1(n) = 0 \\ &Q_2(n)\,Q_1(n)\,f_2(n) = 0 \\ &\vdots \\ &Q_m(n)\,Q_{m-1}(n) \ldots Q_1(n)\,f_m(n) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

Again, it is assumed that $\{f(n)\}_m$ is known, and hence one can solve eqns. 14 for the operators $Q_1(n), Q_2(n), \ldots, Q_m(n)$ [i.e. for $q_1(n), q_2(n), \ldots, q_m(n)$]. It can be shown that eqn. 14 yields the solution

$$q_1(n) = \frac{f_1(n+1)}{f_1(n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

$$q_i(n) = \frac{Q_{i-1}(n+1)\,Q_{i-2}(n+1) \ldots Q_1(n+1)\,f_i(n+1)}{Q_{i-1}(n)\,Q_{i-2}(n) \ldots Q_1(n)\,f_i(n)} \qquad i = 2, 3, \ldots, m \quad (15b)$$

This decomposition method has the advantage (over the first

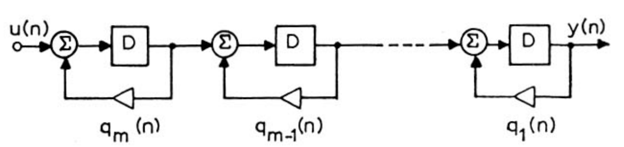

Fig. 3 *Alternative cascade decomposition to that shown in Fig. 2*

method) that it can be applied directly to any desired cascade decomposition, e.g. $L_m(n) = L_{m-p}(n)\,L_p(n)$, where $L_p(n)$ is not necessarily a 1st-order operator.

Relationship between schematic representations: It is possible to obtain the representation of Fig. 3 from that of Fig. 2 by using the equivalence shown in Fig. 4 for every 1st-order operator in the cascade. One then finds [under the condition $r_i(n) \neq 0$; $i = 1, 2, \ldots, m$] the relationships

$$q_1(n) = \frac{r_1(n+1)}{r_1(n)} \qquad q_i(n) = \frac{r_i(n+1)}{r_i(n)}\,q_{i-1}(n+1) \qquad i = 2, 3, \ldots, m \quad (16)$$

It is also observed, from eqns. 1, 11 and 13, that

$$\prod_{i=1}^{m} q_i(n) = \left[\prod_{i=1}^{m+1} r_i(n)\right]^{-1} = a_m(n) \quad . \quad . \quad . \quad . \quad (17)$$

For a given set $\{f(n)\}_m$, one can perform the decomposition by using the functions $f_i(n)$, $i = 1, 2, \ldots, m$ in any order. It is therefore possible, in general, to obtain $m!$ different decompositions. It is clear, however, that eqn. 17 must be satisfied for all possible decompositions.

Example: Consider the 2nd-order operator

$$L_2(n) = E^2 - 2\{(n+1)/(n+2)\}^2 E + \{n/(n+2)\}^2 \qquad (18)$$

It is found that the functions $f_1(n) = 1/n$ and $f_2(n) = 1/n^2$

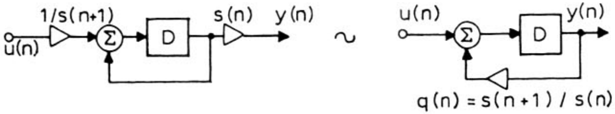

Fig. 4 *Equivalent 1st-order operators* $[s(n) \neq 0]$

satisfy $L_2(n) f_i(n) = 0$; $i = 1, 2$. Applying eqns. 9, 10 and 12, one finds
$r_1(n) = f_1(n) = 1/n$

$$r_2(n) = \Delta \frac{f_2(n)}{r_1(n)} = \Delta \frac{1}{n} = \frac{-1}{n(n+1)} \qquad (19a)$$

$$r_3(n) = 1/\{r_2(n+1)\, r_1(n+2)\} = -(n+1)(n+2) \quad . \quad . \quad (19b)$$

Thus, from eqns. 11,

$$L_2(n)\, y(n) = \frac{1}{(n+2)^2\,(n+1)} \Delta[n(n+1)\, \Delta\{ny(n)\}] \qquad (20)$$

The use of eqn. 15 or eqn. 16 yields an alternative decomposition (the one corresponding to eqn. 13)

$$L_2(n)\, y(n) = \{E - q_1(n)\}\{E - q_2(n)\}\, y(n) \quad . \quad . \quad . \quad (21a)$$

with

$$q_1(n) = n/(n+1) \qquad q_2(n) = n(n+1)/(n+2) \quad . \quad . \quad (21b)$$

The validity of eqn. 17 is easily verified:

$$q_1(n)\, q_2(n) = 1/\{r_1(n)\, r_2(n)\, r_3(n)\} = \{n/(n+2)\}^2 = a_2(n)$$

Conclusions: Two alternative decompositions of a scalar difference operator have been presented and the relationship between them derived. A discrete time-varying system which is specified by its impulse-response function and is describable by eqn. 1 can be realised in a cascade form by applying the above results.

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