Bounds on the Performance of Vector-Quantizers operating under Channel Errors over all Index Assignments

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Abstract

Vector-Quantization (VQ) is an effective and widely implemented method for low-bit-rate communication of speech and image signals. A common assumption in the design of VQ-based communication systems is that the compressed digital information is transmitted through a perfect channel. Under this assumption, quantization distortion is the only factor in output signal fidelity. Moreover, the assignment of channel symbols to the VQ Reconstruction Vectors is of no importance. However, under physical channels, errors may be present, degrading overall system performance. In this case, the effect of channel errors on the VQ system performance depends on the index assignment of the Reconstruction Vectors. For a VQ with N Reconstruction Vectors there are $N!$ possible assignments. Hence, even for relatively small values of N, an exhaustive search over all possible assignments is practically impossible. In this paper, upper and lower bounds on the performance of VQ systems under channel errors over all possible assignments are presented using Linear Programming arguments. These bounds may give the system designer more insight about the gain that could be achieved by improving the index assignment. In numerical examples, the bounds are compared with the performance obtained by using a set of random assignments, as well as with an index assignment obtained by the well-known index switching algorithm.

1. Introduction

Vector Quantization (VQ) is a method for mapping signals into digital sequences [1]. A typical VQ-based communication system is shown in Fig. 1.

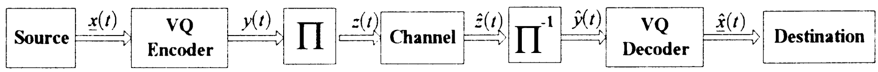

Fig. 1 - Vector Quantization based Communication system

In most Signal Processing applications a discrete-time *Source* emits signal samples over an infinite or a large finite alphabet. These samples should be sent to the *Destination* with the highest possible fidelity. The *VQ Encoder* translates vectors of source samples into *Channel* digital sequences. The task of the *VQ decoder* is to reconstruct source samples from this digital information. Since the analog information cannot be perfectly represented by the digital information some *Quantization Distortion* must be tolerated.

In each channel transmission the VQ encodes a K-dimensional vector of source samples - $\underline{x}(t)$ into a *Reconstruction Vector index* $y(t)$, where the discrete variable t represents the time instant or a channel-use counter. The index is taken from a

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finite alphabet, $y(t) \in \{0,1,\ldots,N-1\}$, where N is the number of Reconstruction vectors (hence the number of possible channel symbols).

The *Index Assignment* is represented in Fig. 1 by a permutation operator:

$$\Pi : y(t) \in \{0,1,\ldots,N-1\} \rightarrow z(t) \in \{0,1,\ldots,N-1\} \tag{1}$$

where a total of $N!$ possible permutations exist. For just 4-bits quantization there are $16! \approx 2 \cdot 10^{13}$ possible permutations. Examination of all possible permutations is therefore impractical. The channel index $z(t)$ is sent through the channel.

For *Memoryless Channels*, The channel output $\hat{z}(t)$ is a random mapping of its input $z(t)$, characterized by the *Channel Probability Matrix* $\mathbf{Q}$, defined by:

$$\{\mathbf{Q}\}_{ij} = \mathrm{Prob}\{\hat{z}(n) = j | z(n) = i\} \tag{2}$$

Throughout we shall assume that $\mathbf{Q}$ is symmetric (i.e., *Symmetric Memoryless Channels*).

For the special case of the *Binary-Symmetric-Channel* (BSC):

$$\{\mathbf{Q}\}_{ij} = \mathrm{Prob}\{\hat{z}(n) = j | z(n) = i\} = q^{H(i,j)}(1-q)^{L-H(i,j)} \tag{3}$$

where L is the number of bits ($N = 2^L$), q is the *Bit-Error-Rate* (BER) and $H(i,j)$ is the *Hamming Distance* between the binary representations of i and j.

At the receiver, after inverse-permutation, the *VQ Decoder* converts the channel output symbols into one of N possible Reconstruction Vectors. It is desired that the Decoder output $\hat{\underline{x}}(t)$ be "close" to the original input. The term "close" will be defined by a distortion measure between the input and the output of the VQ system $d(\underline{x}, \hat{\underline{x}})$.

Knowledge of the source statistics $p(\underline{x})$ or a representing *Training Sequence* is assumed. The perfomance of the overall system is measured in terms of the average distortion $E[d(\underline{x}, \hat{\underline{x}})]$.

In "classic" discussions of VQ applications, the channel is assumed to be noiseless ($\mathbf{Q} = \mathbf{I}$, where $\mathbf{I}$ is the unity matrix), [1], so that no errors occur during transmission and $y(t) = \hat{y}(t)$ for every t. This assumption is based upon using a channel encoder-decoder pair to correct channel errors, causing the distortion due to channel-errors to be negligible. The permutation Π has no effect in this case.

Upon knowledge of the source statistics, Lloyd's algorithm [1] may be used to design the VQ. In practice, a training sequence is used and the LBG algorithm [1] is implemented. Both methods are iterative and alternately apply the *Nearest-Neighbor* Condition and the *Centroid* condition.

In some applications, channel-coding is not utilized due to complexity or Bit-Rate constraints. In case of a channel error event, a wrong Reconstruction Vector is selected at the decoder. The distortion due to channel errors is significant and affects the design of the VQ system [2-12].

The Vector Quantization system consists of a partition of the signal space Ω of all possible input vectors - $\underline{x}$. This space is partitioned into N nonoverlapping regions:

$$\bigcup_i R_i = \Omega; \quad R_i \cap R_j = \varnothing \tag{4}$$

Each partition region R_i has a corresponding *Reconstruction (Representation) Vector* - $\underline{\phi}_i$.

The encoder accumulates a K-dimensional vector of source samples $\underline{x}$. The symbol $y(t) = i$ is emitted if $\underline{x} \in R_i$ and the corresponding channel symbol, $z(t) = \Pi(i)$, is transmitted through the channel. The channel output is a random mapping of this tranmission. Upon receiving the channel symbol $\hat{z}(t) = j$ the decoder emits the Reconstruction Vector - $\underline{\phi}_{\Pi^{-1}(j)}$.

The overall distortion of the VQ-based communication system is:

$$D = E[d(\underline{x}, \hat{\underline{x}})] = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \{\pi \cdot \mathbf{Q} \cdot \pi^T\}_{ij} \int_{R_i} d(\underline{x}, \underline{\phi}_j) \cdot p(\underline{x}) \cdot d\underline{x} \tag{5}$$

In (5) the permutation is represented by a permutation matrix - π, whose entries are 0's and 1's and the sum of each of its rows and columns is 1. For the perfect channel, $\mathbf{Q} = \mathbf{I}$, the permutation matrix π is of no importance ($\pi \cdot \pi^T = \mathbf{I}$), and the only factor of the system performance is the *Quantization Distortion*:

$$D_q = E[d(\underline{x}, \hat{\underline{x}})]\Big|_{\mathbf{Q}=\mathbf{I}} = \sum_{i=0}^{N-1} \int_{R_i} d(\underline{x}, \underline{\phi}_i) \cdot p(\underline{x}) \cdot d\underline{x} \tag{6}$$

For the region R_i, all vectors $\underline{x} \in R_i$ should be represented by $\underline{\phi}_i$. Yet due to channel errors, other reconstruction vectors may appear at the destination. The probability of receving the channels index corresponding to $\underline{\phi}_j$ given the index corresponding to $\underline{\phi}_i$ was transmitted is $\{\pi \cdot \mathbf{Q} \cdot \pi^T\}_{ij}$.

The *Channel Distortion* is defined by the average distance between the reconstructed vector and the one that would have been reconstructed with no channel errors:

$$D_c = \sum_{i=0}^{N-1} p_i \sum_{j=0}^{N-1} \{\pi \cdot \mathbf{Q} \cdot \pi^T\}_{ij} \cdot d(\underline{\phi}_i, \underline{\phi}_j) = trace\{\mathbf{P} \cdot \pi \cdot \mathbf{Q} \cdot \pi^T \cdot \mathbf{D}\} \tag{7}$$

where p_i is the probability that an input vector $\underline{x}$ belongs to the i-th partition region R_i:

$$p_i = \int_{R_i} p(\underline{x}) \cdot d\underline{x} \tag{8}$$

the diagonal matrix $\mathbf{P}$ contains these probabilities $\mathbf{P} = diag\{p_0, p_1, \ldots, p_{N-1}\}$, and the entries of the matrix $\mathbf{D}$ are the distances among all Reconstruction Vectors: $\mathbf{D}_{ij} = d(\underline{\phi}_i, \underline{\phi}_j)$. It is shown in [8],[9] that for the Euclidean distance measure, and Centroid quantizers the overall distortion is the sum of the quantization and channel distortions: $D = D_q + D_c$.

In the literature two main approaches are proposed to improve the performance of Vector Quantizers under channel errors. The first method allows modification of the partition regions and their corresponding codevectors. In the presence of channel errors, and given the transmitted symbol, the received symbol is a random variable. It is suggested to redesign the VQ by modifiying the distortion measure to take all possible output vectors into consideration. This modification results in a *Weighted-Nearest-Neighbor* and *Weighted-Centroid* conditions [7-9]. These conditions are specific to every channel condition. Hence, a VQ designed for a noisy channel should apply a different partition and a different set of codevectors for each possible BER. The main drawbacks of this approach are the large memory consumption and extensive design effort.

The second approach is trying to reduce channel distortion by using a better index assignment. Several suboptimal methods are suggested in the literature. In [7-10] an iterative Index Switching algorithm is proposed. After selecting an initial assignment, the algorithm searchs for a better assignment by exchanging indices of codevectors, and keeping the new assignment if it performs better than its predecessor. This algorithm can only offer a local minima. A more sophisticated algorithm is examined in [7], where Simulated Anealing (SA) is used to search for an optimal index assingnment. The method of SA involves some ad-hoc arguments to define system "temperature" and "cooling" procedures. Moreover, the method of SA has a very slow convergence rate, and cannot assure global optimum during a limited design period.

For the special case of a Uniform Scalar Quantizer with quantization step h, $\phi_i = (i - N/2) \cdot h$; $d(\phi_i, \phi_j) = h^2 (i-j)^2$, and a Uniform Source, $p_i = 1/N$, $i = 0, 1, \ldots, N-1$, it is shown in [2],[5],[6] that the Natural Binary Code Assignment, corresponding here to $\pi = \mathbf{I}$, is the optimal assignment.

The remainder of the paper is organized as follows. In section 2, lower and upper bounds on the performance of VQ system over all possible Index Assignments are presented. In section 3 numerical results are shown. Conclusions are given in Section 4.

2. Performance Bounds

In this section we introduce lower and upper bounds on the channel distortion, as defined in (7), under Symmetric Memoryless Channels, over all possible assignments (permutation matrices - π). The bounding technique is based on eigenvalues and Linear Programming arguments. Instead of optimizing over the (discrete) family of

matrices covering all possible assignments $\pi\mathbf{Q}\pi^T$, we optimize over a wider (continuous) family. A detailed mathematical analysis may be found in [3].
Using the symmetry property of the Channel Matrix, $\mathbf{Q}$, we combine the matrices $\mathbf{D}$ and $\mathbf{P}$ into a single symmetric matrix $\hat{\mathbf{D}} = \mathbf{DP} + \mathbf{P}^T\mathbf{D}^T$. The channel distortion is given then by:

$$D_c = \frac{1}{2} trace\left\{ \mathbf{Q}\pi^T \hat{\mathbf{D}} \pi \right\} \tag{9}$$

Recalling that $\mathbf{Q}$ represents probabilities, the sum of any of its rows is one, so the vector $\underline{1} = [1 \ \ 1 \ \ \cdots \ \ 1]^T$ is an eigenvector of $\mathbf{Q}$ since $\mathbf{Q} \cdot \underline{1} = \underline{1}$.

A fundamental step in the bounding technique is that the matrix $\hat{\mathbf{D}}$ is replaced by another symmetric matrix $\tilde{\mathbf{D}}$, also having $\underline{1}$ as an eigenvector. This replacement changes D_c by a known additive constant. This goal is achieved by adding *"Cross Structured"* matrices, as we define shortly, to the matrix $\hat{\mathbf{D}}$.
We define a *"Column Structured"* matrix as:

$$\mathbf{C}_i = \begin{bmatrix} 0 & & 0 & 1 & 0 & & & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & & 0 & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$
$$\uparrow \ i-\text{th column} \tag{10}$$

Using the property $\mathbf{Q} \cdot \mathbf{C}_i = \mathbf{C}_i$, it is shown in [3] that adding *"Cross Structured"* matrices $\alpha(\mathbf{C}_i + \mathbf{C}_i^T)$, where α is a scalar, changes the r.h.s. of (9) just by the addition of the scalar α, for any permutation matrix π:

$$\frac{1}{2} trace\left\{ \mathbf{Q}\pi^T \left[\hat{\mathbf{D}} + \alpha\left(\mathbf{C}_i^T + \mathbf{C}_i\right) \right] \pi \right\} = \frac{1}{2} trace\left\{ \mathbf{Q}\pi^T \hat{\mathbf{D}} \pi \right\} + \alpha \tag{11}$$

In order to achieve the desired property $\tilde{\mathbf{D}} \cdot \underline{1} = \omega_0 \underline{1}$, for some ω_0, all rows of $\tilde{\mathbf{D}}$ must have the same sum of entries. Let us examine the effect of adding a *"Cross Structured"* matrix $\alpha(\mathbf{C}_i + \mathbf{C}_i^T)$ to a general matrix of size $N \times N$. The sum of all rows except for the i-th row is increased by α, while the sum of the i-th row is increased by $(N+1) \cdot \alpha$.
An algorithm for obtaining $\tilde{\mathbf{D}}$ having the desired property is shown in Table 1. Throughout the algorithm, a variable S is needed to store the sum of all "α" constants added to the r.h.s. of (9). By adding at most N-1 *"Cross Structured"* matrices we get a symmetric matrix where all rows have the same sum of elements, resulting in the matrix $\tilde{\mathbf{D}}$, with the desired property $\tilde{\mathbf{D}} \cdot \underline{1} = \omega_0 \underline{1}$. We shall refer to $\tilde{\mathbf{D}}$ as the *Weighted Distances Matrix*. The channel distortion can now be written as:

$$D_c = \frac{1}{2} trace\left\{ \mathbf{Q}\pi^T \tilde{\mathbf{D}} \pi \right\} - S \tag{12}$$

Initialization: a. Set the matrix: $\tilde{\mathbf{D}} \leftarrow \hat{\mathbf{D}} = \mathbf{DP} + \mathbf{P}^T \mathbf{D}^T$. b. Clear the sum of additive constants: $S \leftarrow 0$. **Step 1**: Calculate the sum of all rows. Denote the sum of the i-th row by $S_i = \sum_{j=0}^{N-1} \left(\tilde{\mathbf{D}}\right)_{ij}$. **Step 2**: Search all rows for the maximal sum of elements. Assume that the row with the maximal sum is labeled k. **Step 3**: For each row $i \neq k$: a. Add the *"Cross Structured"* matrix $\tilde{\mathbf{D}} \leftarrow \tilde{\mathbf{D}} + \frac{1}{N}(S_k - S_i)(\mathbf{C}_i + \mathbf{C}_i^T)$. b. Update $S \leftarrow S + \frac{1}{N}(S_k - S_i)$.

Table 1 - An algorithm for obtaining $\tilde{\mathbf{D}}$ having the property $\tilde{\mathbf{D}} \cdot \underline{1} = \omega_0 \underline{1}$, without affecting the optimization problem

Note that now both the channel matrix $\mathbf{Q}$ and the Weighted Distances Matrix $\tilde{\mathbf{D}}$ are symmetric, have nonnegative entries, and have $\underline{1} = [1 \ \ 1 \ \ \cdots \ \ 1]^T$ as an eigenvector. All eigenvalues of both matrices are real. Next, we use the following Theorem adopted from [14 Section 15.7].

Theorem: The Perron-Frobenius eigenvalue of a nonnegative-entries symmetric matrix $\mathbf{M}$ with the property $\mathbf{M} \cdot \underline{1} = \beta \underline{1}$ is β. This eigenvalue is positive and is the largest in absolute value.

Corollary: The eigenvalue 1 of the matrix $\mathbf{Q}$ and the eigenvalue $\omega_0 > 0$ of the matrix $\tilde{\mathbf{D}}$, both corresponding to the eigenvector $\underline{1}$, are the largest in absolute value for each matrix.

Next, we perform a unitary diagonalization on both matrices:

$$\begin{aligned} &\mathbf{Q} = \mathbf{V} \cdot \Lambda \cdot \mathbf{V}^T \quad && \mathbf{V} \cdot \mathbf{V}^T = \mathbf{I} \\ &\tilde{\mathbf{D}} = \mathbf{W} \cdot \Omega \cdot \mathbf{W}^T \quad && \mathbf{W} \cdot \mathbf{W}^T = \mathbf{I} \end{aligned} \tag{13}$$

Without loss of generality, we arrange the eigenvalues (and their corresponding eigenvectors) in Λ and Ω to be in decreasing order. Substituting (13) into (12):

$$D_c = \frac{1}{2} trace\{\mathbf{V}\Lambda\mathbf{V}^T\pi^T\mathbf{W}\Omega\mathbf{W}^T\pi\} - S = \frac{1}{2} trace\{\Lambda\mathbf{V}^T\pi^T\mathbf{W}\Omega\mathbf{W}^T\pi\mathbf{V}\} - S =$$

$$= \frac{1}{2} trace\{\Lambda\Psi\Omega\Psi^T\} - S = \frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\lambda_i\omega_j\Psi_{ij}^2 - S \tag{14}$$

where we define $\lambda_i = \Lambda_{ii}$, $\omega_i = \Omega_{ii}$, $i = 0,1,\ldots,N-1$ and the matrix Ψ is defined as $\Psi = \mathbf{V}^T\pi^T\mathbf{W}$. The matrix Ψ is also unitary since: $\Psi\Psi^T = \mathbf{V}^T\pi^T\mathbf{W}\mathbf{W}^T\pi\mathbf{V} = \mathbf{I}$.

Since the first column of both $\mathbf{V}$ and $\mathbf{W}$ is $\underline{v}_0 = \underline{w}_0 = \frac{1}{\sqrt{N}}\underline{1}$, and the remaining columns are orthogonal to the vector $\underline{1}$, the structure of $\Psi = \mathbf{V}^T\pi^T\mathbf{W}$ is:

$$\Psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & ? & \\ 0 & & & \end{bmatrix} \tag{15}$$

where the question mark represents unknown entries.

In order to obtain upper and lower bounds over all possible Index Assignments, we relax the constraint that the matrix Ψ equals to $\mathbf{V}^T\pi^T\mathbf{W}$ for some permutation matrix - π (a discrete set of possible Ψ matrices). Instead, we only require the property that the sum of squares of the elements in each row and column of a unitary matrix (Ψ in this case) is equal to 1 (a wider, continuous set of possible Ψ matrices), and state the following optimization problem:

$$\begin{aligned} &\min_{\Psi} / \max_{\Psi} \left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\lambda_i\omega_j\Psi_{ij}^2\right) \\ &s.t. \quad \sum_{i=1}^{N-1}\Psi_{ij}^2 = 1 \quad j = 1,2,\ldots,N-1 \\ &\qquad\;\; \sum_{j=1}^{N-1}\Psi_{ij}^2 = 1 \quad i = 1,2,\ldots,N-1 \end{aligned} \tag{16}$$

Note that the first row and the first column were omitted from the optimization problem. The problem in (16) is a standard Assignment problem in Operations Research, e.g., optimaly assigning N workers to N machines. Using Linear Programming arguments, it is shown in [15] that an optimal solution for the Assignment problem is a permutation matrix Ψ_{opt} that has a single 1 in each row and column, while the remaining elements of the matrix are zero. Nevertheless, Ψ_{opt} does not necessarily correspond to a legal Index Assignment matrix π.

Observing the target function in (16) $\sum_{i=1}^{N-1}\lambda_i\sum_{j=1}^{N-1}\omega_j\Psi_{ij}^2$, we see that the permutation matrix Ψ_{opt} does a one-to-one (permutation) matching between the eigenvalues λ_i and ω_i $i=1,2,\ldots,N-1$, while always matching λ_0 and ω_0.
Recalling that both λ_i and ω_i were arranged in decreasing order, it is shown in [3] that the highest (lowest) possible value is obtained by matching the eigenvalues λ_i and ω_i $i=1,2,\ldots,N-1$ in the same (reversed) order. The minimum and maximum values of the optimization problem are:

Minimum value : $\sum_{i=1}^{N-1}\lambda_i\cdot\omega_{N-i}$ Maximum value : $\sum_{i=1}^{N-1}\lambda_i\cdot\omega_i$

Corresponding to: Corresponding to:

$$\Psi_{min}=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 0 & & & 1\\ \vdots & & & 1 & \\ 0 & & \iddots & & \\ 0 & 1 & & & 0\end{bmatrix} \qquad \Psi_{max}=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & & & 0\\ \vdots & & \ddots & & \\ 0 & & & 1 & \\ 0 & 0 & & & 1\end{bmatrix} \tag{17}$$

and the bounds on Channel Distortion over all possible Index Assignments are:

$$\frac{1}{2}\left(\lambda_0\omega_0+\sum_{i=1}^{N-1}\lambda_i\cdot\omega_{N-i}\right)-S\le D_c\le\frac{1}{2}\left(\lambda_0\omega_0+\sum_{i=1}^{N-1}\lambda_i\cdot\omega_i\right)-S \tag{18}$$

In conclusion, in order to find the desired bounds one should perform the following steps:

1. Calculate the *Weighted Distances Matrix*, $\tilde{\mathbf{D}}$, and the sum of added scalars S, using the algorithm stated in Table 1.
2. Calculate the eigenvalues of the *Channel Matrix* $\mathbf{Q}$ (λ_i, $i=0,1,\ldots,N-1$), and of the *Weighted Distances Matrix* $\tilde{\mathbf{D}}$ (ω_i, $i=0,1,\ldots,N-1$). For the Binary-Symmetric-Channel, λ_i are given in [2], [5].
3. Calculate the upper and lower bounds using (18).

For the special case of an L-bit ($N=2^L$ levels) Uniform Scalar Quantizer and a Uniform Source operating under the Binary Symmetric Channel with Bit-Error-Rate q, these bounds turn out to be [3]:

$$\frac{2(N-1)(N+1)}{3N^2}2q\le D_c\le\frac{2(N-1)(N+1)}{3N^2}\left[1-(1-2q)^L\right] \tag{19}$$

The lower bound coincides with the performance of the Natural Binary Code, which is the optimal Assignment for this case, as shown in [2], [5], [6]. Note that for small Bit Error Rate values, the ratio between the upper and lower bounds in (19) is equal to the number of bits L.

3. Numerical Examples

In this section numerical examples of the performance bounds are presented. Due to the huge number of possible assignments, the lower and upper bounds are compared with the best and worst of 10,000 random assignments. In some cases the performance due to an index assignment obtained by the Index-Switching algorithm [9] is also shown. Further examples may be found in [3].

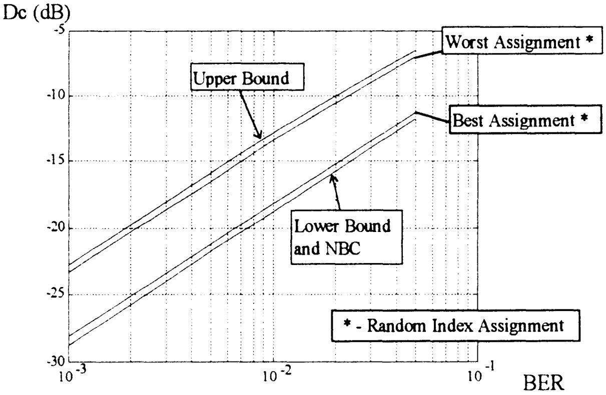

Fig. 2 - Upper and lower bounds over all possible index assignments on the Channel-Distortion of a 4-bit Uniform Scalar Quantizer and a uniform source under the BSC. The bounds are compared with the performance of the best and worst assignments of 10,000 randomly picked assignments. The lower bound coincides with the performance of the optimal assignment (Natural Binary Code)

Example 1: For a 4-bit uniform scalar quantizer, a uniform source and a BSC, bounds were presented in (19). The resulting bounds and the simulation results are shown in Fig. 2. The upper and lower bounds are about 0.5dB away from the best and worst assignments found in the random assignment simulation. As mentioned, The distortion due to the Natural Binary Code (NBC) coincides with the lower bound. The ratio between the upper and lower bound is approximately the number of bits L=4, that is 6dB.

Example 2: Consider the 4-bit uniform scalar quantizer, and the uniform source of Example 1. The digital information is assumed to be sent through the BSC utilizing a (7,4) Hamming Error Correcting Code [16]. The channel transition matrix $\mathbf{Q}$ is symmetric, thus enabling us to use the proposed bounds. The eigenvalues and eigenvectors of $\mathbf{Q}$ are different from the BSC case. The resulting bounds and the simulation results are shown in Fig. 3.
It can be seen that the slope of the graphs is 20dB/ Decade, i.e., reducing the Bit Error Rate by a factor of 10, results in a 20dB lower distortion. The channel distortion is approximately proportional to the square of the Bit Error Rate. The upper bound is about 0.6dB away from the worst random assignment, while the lower bound is about 0.1dB from the best random assignment. It is shown in [3], that the NBC is also optimal for this case. The ratio between the upper and lower

bound is approximately 4.5dB. The addition of the channel protection brought the bounds closer together.

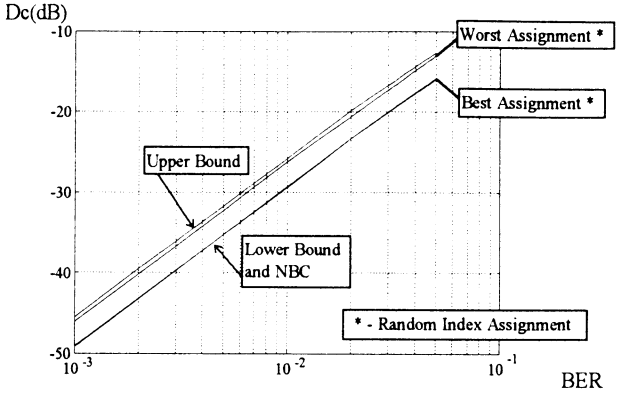

Fig. 3 - Upper and lower bounds over all possible index assignments on the Channel-Distortion of a 4-bit Uniform Scalar Quantizer and a uniform source under the BSC with (7,4) Hamming code. The bounds are compared with the performance of the best and worst assignments of 10,000 randomly picked assignments. The Natural Binary Code coincide with the lower bound.

Example 3: Similar to the first example, we consider now a 4-bit PDF-Optimized Uniform Scalar Quantizer, a Gaussian source and a BSC The resulting bounds and the simulation results are shown in Fig. 4.

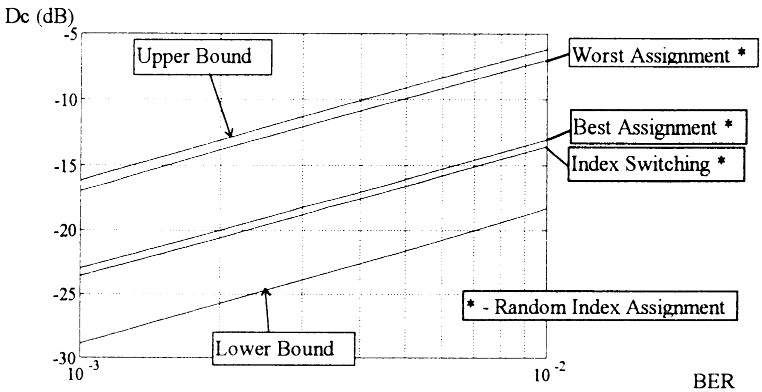

Fig. 4 - Upper and lower bounds over all possible index assignments on the Channel-Distortion of a 4 bit PDF-Optimized Uniform Scalar Quantizer and a Gaussian source under the BSC. The bounds are compared with the performance of the best and worst assignments of 10,000 randomly picked assignments.

The upper bound is about 0.6dB away from the worst assignment found in the random assignment simulation. The lower bound is about 5dB lower than the distortion due to the assignment obtained by the index switching algorithm. As mentioned earlier there are about $2 \cdot 10^{13}$ possible assignments in this example. Since it is not practical to find the best assignment by exhaustive search ,it is not clear at this point how tight the proposed lower bound is. It could well be that the relatively large gap between the lower bound and the performance of the best assignment found in simulations so far, is due to an insuffiecint number of examined assignments (10,000). This issue is presently under investigation.

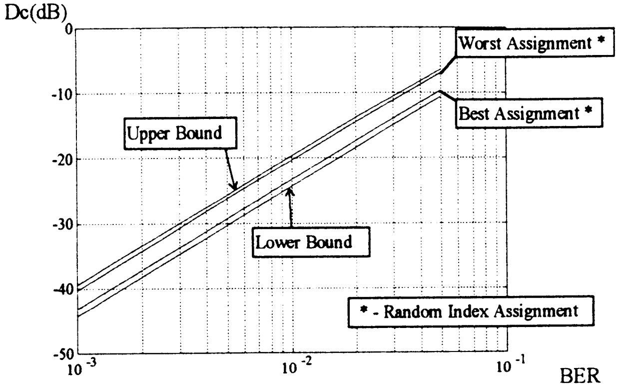

Fig. 5 - Upper and lower bounds over all possible index assignments on the Channel-Distortion of a 4-bit PDF optimized Uniform Scalar Quantizer and a Gaussian source under the Binary Symmetric Channel with (7,4) Hamming code. The bounds are compared with the perforemance of the best and worst assignments of 10,000 randomly picked assignments.

Example 4: Consider the source and quantizer of the previous example. The digital information is sent through a BSC utilizing this time a (7,4) Hamming Error Correcting Code, as in example 2. The resulting bounds and the simulation results are shown in Fig. 5. The upper and lower bounds are about 0.8dB from the best and worst assignment found in the random assignement simulation. As in the case of a uniform source, the addition of channel protection brought the bounds closer together.

As mentioned earlier, further examples may be found in [3]. For a 3-bit PDF-optimized Uniform scalar quantizer, a Gaussian source and the BSC we perform full search over all $8! = 40,320$ possible assignments and the bound appear to be tight. Bounds and simulation result for an 8-bit Vector Quantizer may also be found in [3]

4. Conclusions

In this paper we present upper and lower bounds on the Channel-Distortion for Vector Quantizers operating under channel errors. The bounds were obtained using Linear Programming arguments. Numerical examples are shown for the Binary Symmetric Channel with and without channel Error Correcting Code. For quantizers with 4 bits and more, the bounds are compared with the performance of 10,000 random index assignments. For the Binary Symmetric Channel the upper bounds are close to the performance of the worst assignment found in the random assignment simulation. The lower bounds are sometimes more loose and a significant gap exists between the lower bound and the performance of the assignment obtained by the Index Switching algorithm. This gap may be due to the relatively small number of assignments examined by simulations. This issue is under investigation.

Utilization of an Error Correcting Code decreases the gap between the lower and the upper bounds the gap between the best and the worst assignment found in simulations, and both bounds exhibit a tighter behavior under this conditions.

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